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Breaking waves and persistence property for a two-component Camassa–Holm system

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A R T I C L E I N F O

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A B S T R A C T

In this paper, we study a two-component Camassa–Holm system modeling shallow water waves moving over a linear shear flow. Some new blow-up results and a persistence property in weighted L^p space are established.

1. Introduction

We consider the coupled two-component Camassa–Holm show water system [15,20,28,34,35]

$$\begin{cases} m_t + um_x + 2u_x m - Au_x + \rho\rho_x = 0, & t > 0,\ x \in \mathbb{R}, \\ m = u - u_{xx}, & t > 0,\ x \in \mathbb{R}, \\ \rho_t + (u\rho)_x = 0, & t > 0,\ x \in \mathbb{R}, \end{cases} \tag{1}$$

where the variables $u(t, x)$ and ρ denote the horizontal velocity of the fluid and the free surface elevation from equilibrium respectively, with the boundary assumptions $u \to 0$ and $\rho \to 1$ as $|x| \to \infty$. The parameter $A > 0$ characterizes a linear underlying shear flow propagating in the positive direction of the x-coordinate. All of those are measured in dimensionless units.

System (1) was first derived in [34] and then it has attracted huge amounts of attention since Constantin and Ivanov [15] derived it in the context of shallow water theory and Ivanov [28] gave a rigorous justification of the derivation of system (1), which is a valid approximation to the governing equations for water waves in the shallow water regime with nonzero constant vorticity. System (1) is completely integrable [28], as

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it can be written as a compatibility condition of two linear systems (Lax pair) with a spectral parameter. Moreover, this system admits the following conservation law:

$$\int_{\mathbb{R}} [u^2 + u_x^2 + \rho^2] dx = E(0). \tag{2}$$

Obviously, if $A = \rho = 0$, then system (1) is reduced to the well-known Camassa–Holm (CH) equation, which models the unidirectional propagation of shallow water waves over a flat bottom in [6,16,27,29], as well as water waves moving over an underlying shear flow [30]. The equation also arises in the study of a certain non-Newtonian fluids [5] and also models finite length, small amplitude radial deformation waves in cylindrical hyperelastic rods [17]. The Camassa–Holm equation has many remarkable properties, such as a bi-Hamiltonian structure, Lax completely integrability, infinitely many conservation laws, peakons, wave breaking, etc. Due to its abundant physical and mathematical properties, many physicists and mathematicians pay great attention to this equation [10–14,33]. Among these, numerous impressive works on wave breaking for the Camassa–Holm equation have been obtained, see [7,11,13,14,32,33,36]. Recently, Brandolese and Cortez [2–4], Chen et al. [8,9] introduced new local-in-space blow-up criterions in the study of CH-type and modified equations, which means the condition on the initial data is purely local in space variable. Their works highlight how local structure of solution affects the blow-ups.

For $\rho \not\equiv 0$, System (1) has been studied extensively because it has both solutions which blow up in finite time and solitary wave solutions interacting like solitons. The local well-posedness for the system (1) with initial data $(u_0, \rho_0) \in H^s \times H^{s-1}$, $s \geq 2$ by Kato's semigroup theory was established in [18]. Thereafter, Gui and Liu [20] improved the well-posedness result with initial data in the Besov spaces (especially in $H^s \times H^{s-1}$, $s > \frac{3}{2}$). Meanwhile, the blow-up solutions with a different class of certain initial profiles were shown in [15,18–21]. Since the works [20], we have a good understanding of the behavior of solution to System (1): the only way that a classical solution of System (1) may fail to exist for all time is that the wave may break, which means that the solution remains bounded while its slopes u_x becomes unbounded at a finite time.

For wave-breaking of System (1), we would like to know under what conditions on the initial data $z_0 = (u_0, \rho_0)$, the slope u_x of corresponding solution approaches $-\infty$ in finite time. In this paper, one of our goals is to find some initial data so that at later time u_x^2 grows fast to ∞ along certain line.

Our first wave-breaking result is now formulated as follows.

Theorem 1. *Let $z_0 = (u_0, \rho_0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > \frac{3}{2}$, and T be the maximal existence time of the solution $z = (u, \rho)$ to System (1). Assume that there exists a point $x_0 \in \mathbb{R}$ such that $\rho_0(x_0) = 0$ and*

$$u_{0,x}(x_0) < -\sqrt{2}(|u_0(x_0)| + C_0), \tag{3}$$

where

$$C_0 = \sqrt{\frac{|A|}{4}(E(0) + 3) + \frac{\sqrt{2} - 1}{2} E(0)}. \tag{4}$$

Then the corresponding solution $z = (u, \rho)$ blows up in finite time with an estimate of the blow-up time $\hat{T}$ as

$$\hat{T} \leq \frac{1}{\sqrt{2}C_0} \log\Big(\frac{\sqrt{u_{0,x}^2(x_0) - 2u_0^2(x_0)} + \sqrt{2}C_0}{\sqrt{u_{0,x}^2(x_0) - 2u_0^2(x_0)} - \sqrt{2}C_0}\Big).$$