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Dynamical properties of weighted composition operators on the space of smooth functions

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ABSTRACT

The aim of this paper is to investigate several dynamical properties of weighted composition operators acting on the space of smooth functions. First we characterize hypercyclic, weakly mixing and mixing weighted composition operators. As a by-product we obtain a characterization of hypercyclic, weakly mixing and mixing composition operators. Then we investigate the special one-dimensional case.

1. Introduction

Let $\mathbb{K}$ be the field of real or complex numbers, let $\Omega \subset \mathbb{R}^d$ be open and let $C^\infty(\Omega, \mathbb{K})$ be the space of $\mathbb{K}$ valued smooth functions on Ω equipped with the standard topology of uniform convergence of functions and all their partial derivatives on compact sets. This topology is generated by the family

$$\{||\cdot||_{K,n} : K \subset \Omega \text{ compact}, n \in \mathbb{N}\}$$

of seminorms, where

$$||f||_{K,n} = \max_{x \in K} \max_{|\alpha| \le n} \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right|.$$

The aim of this paper is to investigate several dynamical properties of composition operators and weighted composition operators acting on this space, i.e., operators of the form

$$C_\psi : C^\infty(\Omega, \mathbb{K}) \to C^\infty(\Omega, \mathbb{K}), \;\; F \mapsto F \circ \psi$$

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and

$$C_{w,\psi} : C^\infty(\Omega, \mathbb{K}) \to C^\infty(\Omega, \mathbb{K}),\ F \mapsto w \cdot (F \circ \psi),$$

where $\psi : \Omega \to \Omega$ and $w : \Omega \to \mathbb{K}$ are smooth. We are interested in characterizing when these operators are hypercyclic, weakly mixing or mixing. Let us briefly recall these concepts. So let X be a Fréchet space and let $T : X \to X$ be an operator (i.e., linear and continuous). For every $n \geq 1$ the operator $T^n : X \to X$ is defined as the n-th iterate of T, i.e.,

$$T^n = \underbrace{T \circ \cdots \circ T}_{n\ times}.$$

The operator $T : X \to X$ is called hypercyclic if there exists $x \in X$ such that the set

$$\mathrm{orb}(x, T) := \{T^n(x) : n \geq 1\}$$

is dense in X. Such an x is called a hypercyclic vector of T. By the famous Birkhoff's Transitivity Theorem, an operator T acting on a Fréchet space X is hypercyclic if and only if it is topologically transitive, i.e., for every two nonempty open sets $U, V \subset X$ there is $n \in \mathbb{N}$ such that $T^n(U) \cap \mathrm{V} \neq \emptyset$. The operator T is called weakly mixing if the operator $T \times T : X \times X \to X \times X$ is topologically transitive, i.e., for every four nonempty open sets $U_1, U_2, V_1, V_2 \subset X$ there is $n \in \mathbb{N}$ such that $T^n(U_1) \cap V_1 \neq \emptyset$ and $T^n(U_2) \cap V_2 \neq \emptyset$. Finally, T is called mixing if for every two nonempty open sets $U, V \subset X$ there is $N \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ for every $n \geq N$. From the very definitions it is clear that every mixing operator is weakly mixing and every weakly mixing operator is hypercyclic. Let us note that the first example of an operator which is hypercyclic but not weakly mixing was given by de la Rosa and Read (see [5]), an example of an operator which is weakly mixing but not mixing can be found in [7, Ex. 3.11]. For a great exposition of the subject of linear dynamics we refer to two recently published monographs [1,7].

In this paper we characterize these weighted composition operators acting on the space of smooth functions which are hypercyclic, weakly mixing or mixing (see Theorem 3.5 and Theorem 3.6). It turns out that the dynamical properties of the operator $C_{w,\psi}$ do not depend on the topological properties of the set Ω, the role played by the weight function w in this context is not so important, and that the dynamical properties of $C_{w,\psi}$ are heavily related with the dynamical properties of the function ψ. As an immediate consequence of our results we obtain a characterization of these composition operators acting on the space of smooth functions which are hypercyclic, weakly mixing or mixing (see Corollary 3.7 and Corollary 3.8). Let us also note that minor modifications of the proofs of the statements mentioned above lead to characterization of dynamical properties of (weighted) composition operators acting on the space $C(\Omega, \mathbb{K})$ of all $\mathbb{K}$ valued continuous functions on Ω, equipped with the topology of uniform convergence on compact sets (see Theorem 3.9 and Theorem 3.10). In the last section of this paper we show that all the dynamical properties of the operator $C_{w,\psi}$ are equivalent in case if $\Omega = \mathbb{R}$.

There is a huge literature devoted to the study of dynamical properties of (weighted) composition operators acting on various spaces of functions. Let us note that one of the first examples of a hypercyclic operator was given by Birkoff (see [2]) who proved that the translation operator (which is a composition operator) acting on the space of entire functions on the complex plain is hypercyclic. Later on Grosse-Erdmann and Mortini (see [6]) characterized hypercyclic composition operators acting on the space of holomorphic functions $H(\Omega)$, where $\Omega \subset \mathbb{C}$ is open (in this case the dynamical properties of these composition operators heavily rely on the topological properties of the set Ω). Zając in [11] obtained a characterization of hypercyclic composition operators acting on $H(\Omega)$, where $\Omega \subset \mathbb{C}^d$ is a domain of holomorphy, or if it is even a Stein manifold. In [3], Bonet and Domański obtained, among other results, a characterization of topologically transitive composition operators acting on the space of real analytic functions. There are results about