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Note

A note on the relationship between quasi-symmetric mappings and φ-uniform domains

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ABSTRACT

The aim of this note is to construct a ψ-uniform domain G in the complex plane $\mathbb{C}$ such that the identity mapping id: $(G, j_G) \to (G, k_G)$ is not an η-quasi-symmetric mapping for any homeomorphism η: $[0, \infty) \to [0, \infty)$. This result shows that the answer to the related open problem, posed by Hästö, Klén, Sahoo and Vuorinen, is negative.

1. Introduction

For a proper subdomain G of $\mathbb{R}^n$ and z_1, $z_2 \in G$, the distance ratio metric j_G is defined by

$$j_G(z_1, z_2) = \log\left(1 + \frac{|z_1 - z_2|}{\min\{\delta_G(z_1), \delta_G(z_2)\}}\right),$$

where $\delta_G(z_1)$ denotes the Euclidean distance from z_1 to the boundary ∂G of G. We remark that the above form of j_G, introduced in [10], is obtained by a slight modification of a metric that was studied in [2,3].

For a rectifiable arc or a path γ in G, its *quasihyperbolic length* of γ in G is the number:

$$\ell_{k_G}(\gamma) = \int_\gamma \frac{|dz|}{\delta_G(z)}.$$

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The *quasihyperbolic metric* $k_G(z_1, z_2)$ between z_1 and z_2 is defined by

$$k_G(z_1, z_2) = \inf\{\ell_{k_G}(\gamma)\},$$

where the infimum is taken over all rectifiable arcs γ joining z_1 and z_2 in G. It is well-known that for z_1 and $z_2 \in G$, we have $k_G(z_1, z_2) \geq j_G(z_1, z_2)$ (cf. [3]).

The class of uniform domains was introduced by Martio and Sarvas in 1979 [6]. The precise definition is as follows.

Definition 1.1. Given $c \geq 1$, a domain G in $\mathbb{R}^n$ is called *c-uniform* provided that each pair of points z_1, z_2 in G can be joined by a rectifiable arc γ in G satisfying

(1) $\min\{\ell(\gamma[z_1, z]), \ell(\gamma[z_2, z])\} \leq c\, \delta_G(z)$ for all $z \in \gamma$;
(2) $\ell(\gamma) \leq c|z_1 - z_2|$,

where $\ell(\gamma)$ denotes the length of γ and $\gamma[z_j, z]$ stands for the part of γ between z_j and z. An arc γ with the above properties is called a *double c-cone* arc. A domain is called *uniform* if it is c-uniform for some constant $c \geq 1$.

The following convenient characterization of uniform domains, by means of the quasihyperbolic and distance ratio metrics, was given by Gehring and Osgood [2]: a proper subdomain G of $\mathbb{R}^n$ is uniform if and only if there exists a constant $\mu \geq 1$, depending only on c, such that for all z_1 and z_2 in G,

$$k_G(z_1, z_2) \leq \mu j_G(z_1, z_2).$$

We remark that the above characterization is again slightly different from the one given in [2], as the original result had an additive constant on the right hand side. Later, it was shown by Vuorinen [10] that this constant is not necessary. Motivated by this observation, Vuorinen [10] gave the following more general definition of *φ-uniform* domains:

Definition 1.2. Let φ: $[0, \infty) \to [0, \infty)$ be a homeomorphism. A domain $G \subset \mathbb{R}^n$ is said to be *φ-uniform* if for all z_1, $z_2 \in G$,

$$k_G(z_1, z_2) \leq \varphi\Big(\frac{|z_1 - z_2|}{\min \delta_G(z_1), \delta_G(z_2)}\Big).$$

Obviously, uniformity implies φ-uniformity with $\varphi(t) = \mu \log(1 + t)$ for $t > 0$ with $\mu \geq 1$. It is easy to see that the converse is not true.

Interesting results on the above classes of domains have been obtained by Väisälä [7] (see also [8]). In particular, he observed that the class of φ-uniform domains coincides with the class of uniform domains if φ is a *slow function*, i.e.,

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = 0.$$

Recently, the geometric properties of this class of domains have been investigated in [4]. The stability of φ-uniform domains has been established [5].

Definition 1.3. A homeomorphism f: $\mathbb{R}^n \to \mathbb{R}^n$ is said to be *η-quasi-symmetric* if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

$$|x - a| \leq t|x - b| \text{ implies } |f(x) - f(a)| \leq \eta(t)|f(x) - f(b)|$$

for each $t > 0$ and for all points x, a and b in $\mathbb{R}^n$.