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Estimates for Weierstrass division in ultradifferentiable classes

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ABSTRACT

We study the Weierstrass division theorem for function germs in strongly non-quasianalytic Denjoy–Carleman classes $\mathcal{C}_M$. For suitable divisors $P(x,t) = x^d + a_1(t)x^{d-1} + \cdots + a_d(t)$ with real-analytic coefficients a_j, we show that the quotient and the remainder can be chosen of class $\mathcal{C}_{M^\sigma}$, where $M^\sigma = ((M_j)^\sigma)_{j\geq 0}$ and σ is a certain Łojasiewicz exponent related to the geometry of the roots of P and verifying $1 \leq \sigma \leq d$. We provide various examples for which σ is optimal, in particular strictly less than d, which sharpens earlier results of Bronshtein and of Chaumat–Chollet.

0. Introduction

Consider a polynomial P of degree d given by

$$P(x,t) = x^d + a_1(t)x^{d-1} + \cdots + a_d(t)$$

where $a_1, \ldots, a_d$ are real-analytic function germs at the origin in $\mathbb{R}^m$ such that $a_j(0) = 0$ for $j = 1, \ldots, d$. The classic Weierstrass division theorem states that for any real-analytic function germ f at the origin in $\mathbb{R} \times \mathbb{R}^m$, we have a unique division formula

$$f(x,t) = P(x,t)q(x,t) + \sum_{j=0}^{d-1} r_j(t)x^j,$$

where q and $r_0, \ldots, r_{d-1}$ are real-analytic function germs at the origin in $\mathbb{R} \times \mathbb{R}^m$ and $\mathbb{R}^m$, respectively. The famous Malgrange–Mather division theorem [10,11,17] consists in a similar statement in the $\mathcal{C}^\infty$ setting: if f and $a_1, \ldots, a_d$ are $\mathcal{C}^\infty$ function germs, the division formula still holds with $\mathcal{C}^\infty$ function germs q and $r_0, \ldots, r_{d-1}$, which are no longer unique.

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In view of these two fundamental results, it is natural to ask what kind of division results can be obtained for classes of functions which are "between analytic and $\mathcal{C}^\infty$", namely Denjoy–Carleman classes $\mathcal{C}_M$ associated with a sufficiently regular sequence M of real numbers (in particular, the stability of $\mathcal{C}_M$ classes under derivation will always be implied). If f and the $a_1, \ldots, a_d$ are of class $\mathcal{C}_M$, is it possible to achieve Weierstrass division with q and $r_0, \ldots, r_{d-1}$ in the same class, or maybe in some larger class $\mathcal{C}_N$ depending on M and on the polynomial P? In the absence of extra assumptions, it quickly turns out that, in general, one cannot expect that the quotient and the remainder will be in the same class as f, even if the coefficients $a_1, \ldots, a_d$ are real-analytic: see, for instance, Proposition 2 in Section 3 of [16].

When the class $\mathcal{C}_M$ is quasianalytic, it was proved by Childress [8] that in order to have a division property without loss of regularity, that is, with the quotient and the remainder in the same class $\mathcal{C}_M$ as f, it is necessary that P be hyperbolic (this condition means that for each value of the parameter t, all the roots of $P(\cdot, t)$ are real). Interestingly, it turns out that hyperbolicity is also a sufficient condition for division without loss of regularity, as shown later by Chaumat–Chollet [7].

In the present paper, instead of quasianalytic classes, we shall concentrate on the case of so-called *strongly regular* (in particular, non-quasianalytic) classes, a typical example of which is provided by Gevrey classes $\mathcal{G}^{1+\alpha}$ associated with $M_j = (j!)^\alpha$ for some real $\alpha > 0$. A first important result in this direction is due to Bronshtein [3], who proved that for $\mathcal{G}^{1+\alpha}$ data, the quotient and the remainder can be chosen of class $\mathcal{G}^{1+d\alpha}$. This was later considerably extended by Chaumat–Chollet [6], who showed in particular that if f and $a_1, \ldots, a_d$ belong to a strongly regular class $\mathcal{C}_M$, the quotient and remainder can be chosen of class $\mathcal{C}_N$ with $N_j = M_{dj}$, or equivalently, $N_j = (M_j)^d$. This result appears in [6] as a corollary of a Weierstrass division theorem for generic polynomials $\Pi(x, \lambda) = x^d + \lambda_1 x^{d-1} + \cdots + \lambda_d$, using the fact that $P(x, t) = \Pi(x, a_1(t), \ldots, a_d(t))$. Since the exponent d in $(M_j)^d$ is actually the best possible for the generic polynomial of degree d, this approach is unlikely to provide better estimates taking into account particular geometric or algebraic features of a given Weierstrass polynomial P.

In the case of a hyperbolic polynomial P, using a combination of more direct proofs of Weierstrass division and of specific information on the regularity of the roots of P, it is in fact possible to obtain $N_j = M_j$: this was proved by Bronshtein [3] for Gevrey classes, and by Chaumat–Chollet in the general case, as the article [7] actually encompasses both quasianalytic and non-quasianalytic situations.

This suggests that it should be also possible to improve the general estimate $N_j = (M_j)^d$ for suitable non-hyperbolic polynomials P, in terms of algebraic or geometric features of P. In this article, we shall indeed obtain such an improvement, provided P is real-analytic and the locus Γ of its complex roots has a sufficiently well-behaved geometry, in a sense described precisely in Section 2. Although this is only a partial answer, it covers a number of standard examples. We shall get $N_j = (M_j)^\sigma$ where σ is a rational number which depends on P and satisfies $1 \leq \sigma \leq d$. This number is introduced as a Łojasiewicz exponent associated with metric properties of Γ and of the sets $\mathcal{N}_z$ of complex parameters τ such that $P(z, \tau) = 0$. For instance, for $m = 1$ and $P(x, t) = x^d - t^2$, we have $\sigma = d/2$ and our result therefore shows that if the function germ f is of Gevrey class $\mathcal{G}^{1+\alpha}$, the quotient and the remainder for the Weierstrass division of f by P can be chosen with $\mathcal{G}^{1+\frac{d}{2}\alpha}$ regularity, instead of the cruder $\mathcal{G}^{1+d\alpha}$ estimate given by previously known results. We shall also see that the improved estimates are actually optimal: for certain germs f, the quotient and remainder may not be chosen in any class $\mathcal{C}_N$ smaller than $\mathcal{C}_{M^\sigma}$.

The paper is organized as follows. In Section 1, we recall the necessary definitions and fundamental facts pertaining to Denjoy–Carleman classes. In Section 2, we gather all the geometric material that will be used, introducing in particular the assumptions on P, and the aforementioned Łojasiewicz exponent σ. Several examples are provided to illustrate the assumptions and explicit computations of σ. Section 3 is devoted to results on $\bar{\partial}$-flat extensions of functions: these results are a technical tool in our proof of the division formula with improved estimates, which is eventually carried out, and discussed, in Section 4.