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On affine translation surfaces in affine space

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ABSTRACT

In this work we give a systemic study of affine translation surfaces in affine 3-dimensional space. Specifically, we obtain the complete classification of minimal affine translation surfaces. Moreover, we consider affine translation surfaces with some natural geometric conditions, such as constant affine mean curvature and constant Gauss–Kronecker curvature. Some characterization results with these geometric conditions are also obtained.

1. Introduction

In 3-dimensional Euclidean space $\mathbb{E}^3$, a surface is called a translation surface if it is a graph of a function with the form

$$F(x, y) = f(x) + g(y),$$

where $f(x)$ and $g(y)$ are differentiable functions. The study of translation surfaces could be traced back to Scherk in 1835, who proved that, besides the plane, the only minimal translation surface is the surfaces (so-called Scherk surface) given by

$$F(x, y) = \frac{1}{a} \ln \left| \frac{\cos(ax)}{\cos(ay)} \right|,$$

where a is nonzero constant. Since then, the geometric properties of translation surfaces have been intensively studied by many geometers. For example, Liu in [4] classified translation surfaces with constant mean curvature and constant Gauss curvature in Euclidean 3-space.

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In the literature of affine differential geometry, translation surfaces have been also studied intensively. Initially, minimal translation surfaces were studied by Manhart [7] and Verstraelen–Vrancken [11] independently. In particular, in 1985 Manhart [7] gave a complete explicit classification of nondegenerate minimal translation surfaces in affine space $\mathbb{A}^3$. Furthermore, Sun [9] classified translation surface with nonzero constant mean curvature in affine space $\mathbb{A}^3$. The second author and Hou [2] considered affine translation surfaces with constant Gauss–Kronecker curvature in $\mathbb{A}^3$. Recently, Yang, Yu and Liu [12] gave some classification results for nondegenerate linear Weingarten centroaffine translation surfaces in affine space $\mathbb{A}^3$.

The study of translation surfaces were also extended in affine spaces with higher dimensions or higher codimensions, see Magid–Vrancken and Sun–Chen's works [6,10].

Very recently, Liu and Yu in [5] generalized the definition of translation surfaces in Euclidean spaces. They defined an *affine translation surface* by a graph of a function as follows:

$$F(x,y) = f(x) + g(ax + y)$$

for some nonzero constant a and differential functions f, g. Note that in the case $a = 0$, the *affine translation surface* becomes the classical translation surface. Interestingly, Liu and Yu proved that, besides the plane, the only minimal affine translation surface in Euclidean 3-space $\mathbb{E}^3$ is the surface given by

$$F(x,y) = \frac{1}{c} \ln \left| \frac{\cos(c\sqrt{1+a^2}x)}{\cos[c(ax+y)]} \right|,$$

where a, c are constant and $ac \neq 0$.

Motivated by the nice work mentioned above, in this paper we consider the geometry of affine translation surfaces in affine space $\mathbb{A}^3$. The most interesting thing is to classify all minimal affine translation surfaces. It's well known that affine differential geometry is quite different from Euclidean geometry, see [1,3,8]. We first get the following classification theorem for minimal affine translation surfaces in $\mathbb{A}^3$.

Theorem 1.1. *Let M be a nondegenerate minimal affine translation surface in $\mathbb{A}^3$. Then M is affinely equivalent to one of the graphs of the following eight functions:*

$$z = x^2 \pm (ax + y)^2; \tag{1.1}$$

$$z = x^2 \pm (ax + y)^{2/3}; \tag{1.2}$$

$$z = x^{2/3} \pm (ax + y)^2; \tag{1.3}$$

$$z = x^{2/3} \pm (ax + y)^{2/3}; \tag{1.4}$$

$$z = \pm \ln x \pm (1 - \cos s), \quad s - \sin s = ax + y; \tag{1.5}$$

$$z = \pm(1 - \cos t) \pm \ln(ax + y), \quad t - \sin t = x; \tag{1.6}$$

$$z = \pm(1 + \cosh t) \pm (1 - \cos s), \quad t + \sinh t = x,\ s - \sin s = ax + y; \tag{1.7}$$

$$z = \pm(1 - \cos t) \pm (1 + \cosh s), \quad t - \sin t = x,\ s + \sinh s = ax + y. \tag{1.8}$$

Remark that our current results generalize Manhart's results [7]. All these surfaces obtained in Theorem 1.1 are new minimal surfaces in affine spaces.

As an application of Theorem 1.1, we reprove Manhart's results [7] for minimal translation surfaces as follows:

Corollary 1.2. *Let M be a nondegenerate minimal translation surface in $\mathbb{A}^3$. Then M is affinely equivalent to one of the graphs of the following five functions:*