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On a nonlocal extension of differentiation

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We study an integral equation that extends the problem of anti-differentiation. We formulate this equation by replacing the classical derivative with a known nonlocal operator similar to those applied in fracture mechanics and nonlocal diffusion. We show that this operator converges weakly to the classical derivative as a nonlocality parameter vanishes. Using Fourier transforms, we find the general solution to the integral equation. We show that the nonlocal antiderivative involves an infinite dimensional set of functions in addition to an arbitrary constant. However, these functions converge weakly to zero as the nonlocality parameter vanishes. For special types of integral kernels, we show that the nonlocal antiderivative weakly converges to its classical counterpart as the nonlocality parameter vanishes.

1. Introduction

We consider integral equations for distributions u_ϵ on $\mathbb{R}$ of the form:

$$D_{\alpha,\epsilon}u_\epsilon(t) := -\alpha_\epsilon * u_\epsilon(t) = F(t), \quad t \in \mathbb{R}, \tag{1.1}$$

where $*$ denotes the convolution, $f * g\,(t) = \int_{\mathbb{R}} f(t-s)g(s)\,\mathrm{d}s$, α_ϵ is an anti-symmetric function on $\mathbb{R}$ that depends on a positive "nonlocality parameter" ϵ, and $D_{\alpha,\epsilon}$ is a "nonlocal derivative".

Nonlocality describes interactions over distances; in (1.1), it refers to the fact that $F(t)$ is related to $u(s)$ via D for values of s far from t, where "far" is quantified by the parameter ϵ. Mathematically speaking, D_{α_ϵ} is "strongly nonlocal" in the sense of Rogula [18], since the support of $D_{\alpha,\epsilon}u$ is not contained in that of u. In contrast, the derivative $u'(t)$ does satisfy this property, so it is a "local" operator (it is also "weakly nonlocal"; see [18]).

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We define $D_{\alpha,\epsilon}$ in such a way that $D_{\alpha,\epsilon}f(t) \to f'(t)$ in some sense as $\epsilon \to 0$. In other words, we think of (1.1) as a "nonlocal extension" of the first order ordinary differential equation (ODE):

$$u'(t) = F(t), \quad t \in \mathbb{R}. \tag{1.2}$$

We solve (1.1) using the Fourier transform:

$$\hat{u}(\xi) = \mathcal{F}[u](\xi) = \int_{-\infty}^{\infty} e^{-2\pi i\xi t} u(t)\mathrm{d}t,$$
$$u(t) = \mathcal{F}^{-1}[\hat{u}](t) = \int_{-\infty}^{\infty} e^{2\pi it\xi} \hat{u}(\xi)\mathrm{d}\xi. \tag{1.3}$$

Nonlocal models, though well known since the 1800's viz. fractional derivatives, have recently found successful application in Silling's [22] theory of peridynamic fractures; for applications of these nonlocal operators in nonlocal diffusion and image processing, see [2,10].

The success of nonlocal models stems from the ease with which they handle singularities. Although classical models using differential equations cannot well describe discontinuous solutions to these equations, such as those occurring in fracture dynamics, the integral operators applied by Silling suffer no such difficulties. Using nonlocal operators in these classical models extends the types of physical processes that we can model and the types of qualitative behavior that we can describe.

An important feature of these operators is that they are extensions of classical differential operators. In [6], it is shown that such first order nonlocal operators converge strongly in L^2 to classical partial derivatives as a nonlocality parameter vanishes. This extensivity property is important since it allows us to preserve much of the physical structure that makes up classical differential models. In addition, when the classical descriptions are correct, the nonlocal frameworks can recover these results by passing to the "classical limits".

The majority of the nonlocal literature has focused on so-called second order models. These nonlocal models extend second order partial differential equations and boundary value problems, such as the wave equation [25], Laplace's equation [15] and the heat equation [1] (see also the nonlocal counterpart to the fourth order biharmonic equation [17]). However, the literature for nonlocal extensions of first order models is more sparse. Du et al. [7] studied a nonlocal extension of the nonlinear advection equation. Among other results, they showed that inviscid solutions to this nonlinear equation do not blow up in finite time, a stark contrast to those of the classical equation. There is also some literature available for first order models using fractional derivatives (see [12] for a large set of examples).

Given the success of these nonlocal operators in extending second order models, we are interested in their application to first order models, specifically first order ordinary differential equations (ODEs). First order models are ubiquitous in the natural sciences. One area that is currently under active research is the treatment of discontinuous models. These are ODEs that have discontinuous "forcing terms", and have applications in the flow through porous soil [9], static friction problems [23], and optimal control with discrete feedbacks [14]. A challenge with classical (i.e. differential) frameworks is to solve these problems numerically [4], since the classical derivatives for solutions to these problems do not exist everywhere. As a result, many heuristic and complicated numerical methods are needed to solve these equations computationally. Although we do not pursue this here, one motivation for studying nonlocal first order models is the possibility of replacing classical derivatives in these equations with nonlocal derivatives. It is possible that solving discontinuous integral equations, which do not contain classical derivatives, is a more straightforward task than solving discontinuous ODEs.

The contributions of this paper are as follows.