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An irreversible investment problem with maintenance expenditure on a finite horizon: Free boundary analysis ☆

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ABSTRACT

This paper concerns a continuous-time, finite horizon, optimal irreversible investment problem with maintenance expenditure of a firm under uncertainty. We assume that the firm can make irreversible investments to expand its production capacity and spend maintenance expenditure to achieve better performance of the productivity. The objective of the firm is to construct optimal investment and maintenance policies to maximize its expected total profit over a finite horizon. Mathematically, it is a singular stochastic control problem whose value function satisfies a parabolic variational inequality with gradient constraint. The problem gives rise to two free boundaries which stand for the optimal investment and maintenance strategies, respectively. We investigate behaviors of free boundaries, study regularities of the value function, and give optimal investment and maintenance policies. As we know, this is a first integral result for an investment–maintenance problem with a finite time horizon due to use of partial differential equation (PDE) technique.

1. Introduction

This paper concerns a continuous-time, finite horizon, optimal irreversible investment problem with maintenance expenditure of a firm under uncertainty. Optimal investment problems have been studied widely in the last years. Bertola [1] considered an optimal investment problem under uncertainty of a firm and characterized the optimal investment–disinvestment policy. Dai and Yi [5,6] concerned an optimal investment problem with transaction costs on a finite time. In [5,6], the behaviors of the free boundary is characterized and the regularity of the value function is proved. Several authors studied the firm's optimal

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problem of irreversible investment (see [4,19,17,3,7]). In particular, Chiarolla and Ferrari [3,7] derived a new integral equation for the free boundary of the infinite and finite horizon, respectively.

Yet there is an increasing effort to incorporate maintenance in the core of investment theory. The incorporation of maintenance cost in macroeconomic models of investment has truly started with the illuminating work of McGrattan and Schmitz [15]. They are the first to exploit a Canadian survey and to highlight why and how investment theory can account for these costs. Since then, several research projects have been launched on the topic. Kalaitzidakis and Kalyvitis [10,11] considered how maintenance of public capital affects long-term growth and how to fix optimal maintenance expenditures. In [2], the authors studied investment and maintenance co-movements without any postulated special depreciation function. One paper related to the present work is Kawaguchi and Morimoto [12] where the authors concerned an infinite horizon investment problem with maintenance expenditure of a firm under uncertainty. They achieved the optimal policy and proved the existence of the optimal investment boundary point, but did not consider the optimal maintenance boundary.

In this paper, we concern the model of [12] with a finite horizon, and aim to provide a theoretical analysis of behaviors of optimal investment and maintenance boundaries, respectively. It is challenging to take a finite horizon case into consideration since the corresponding free boundaries (optimal policies) vary with time. We attack the problem by virtue of a PDE approach.

Mathematically, it is a singular stochastic control problem whose value function satisfies a parabolic variational inequality with gradient constraint. The problem gives rise to two free boundaries which stand for the optimal investment and maintenance strategies, respectively. The main task is to characterize behaviors of the two free boundaries. But it is not an easy task. First, the optimal investment boundary lies between the domain $\{(z,\tau) \in \Omega_T : \partial_z u(z,\tau) = p_2 e^z\}$ and the domain $\{(z,\tau) \in \Omega_T : \partial_z u(z,\tau) < p_2 e^z\}$, and $u(z,\tau)$ satisfies a variational inequality with gradient constraint. However, it is intractable to study the free boundary from the original variational inequality with gradient constraint. So we intend to reduce the original problem to a standard variational inequality with function constraint, but the variational inequality with function constraint is not a self-contained system, which leads to difficulties to construct a connection between the above two variational inequalities. Secondly, the optimal maintenance boundary is the boundary between the region $\{(z,\tau) \in \Omega_T : u(z,\tau) - \partial_z u(z,\tau) - p_1 > 0\}$ and the region $\{(z,\tau) \in \Omega_T : u(z,\tau) - \partial_z u(z,\tau) - p_1 < 0\}$. That is, it is a level set of $\{(z,\tau) \in \Omega_T : u(z,\tau) - \partial_z u(z,\tau) - p_1 = 0\}$, which is different to the free boundary with function constraint or with gradient constraint. To the best of our knowledge this is a complete novelty in the literature on singular stochastic control problems with a finite horizon.

The paper is organized as follows. In section 2, we present the model formulation. Section 3 is devoted to studying regularities of solution to problem (2.9) with a known $u(z,\tau)$. In section 4, we exploit the auxiliary condition with which problem (2.9) can be transformed the self-contained Problem A and prove that the Problem A has a solution by the Schauder fixed point theorem. In addition, we obtain a classical solution to problem (2.7) and construct a connection between problem (2.7) and (2.9). The behaviors of the optimal investment and maintenance boundaries are investigated in Section 5 and section 6, respectively. We give the optimal investment and maintenance policies in Section 7. Section 8 concludes the paper.

2. Problem formulation

We consider an optimal investment problem with maintenance expenditure of a firm under uncertainty with a finite horizon. As in [12], the firm faces uncertain future changes in the productivity of the capital stock, which are modeled by diffusion processes, and makes irreversible investments in capital goods. The maintenance expenditure can improve the productivity of the existing capital stock. We assume T is maturity time. For any time $0 \le t \le T$, the measure of capital productivity X_s of the firm and the capital stock Y_s at time $s \ge t$ are given by the stochastic differential equation