J. Math. Anal. Appl. 440 (2016) 677–691

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the Dirichlet problem in billiard spaces

Grzegorz Gabor

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland

ARTICLE INFO

Article history:
Received 29 May 2015
Available online 29 March 2016
Submitted by M. Quincampoix

Keywords:
Dirichlet problem
State-dependent impulses
BVP
Billiard

ABSTRACT

The constrained Dirichlet boundary value problem $\ddot{x} = f(t, x)$, $x(0) = x(T)$, is studied in billiard spaces, where impacts occur in boundary points. Therefore we develop the research on impulsive Dirichlet problems with state-dependent impulses. Inspiring simple examples lead to an approach enabling to obtain both the existence and multiplicity results in one dimensional billiards. Several observations concerning the multidimensional case are also given.

1. Introduction

For three hundred and fifty years several important problems of mechanics and physics have been investigated and modeled as mathematical problems with impacts. Starting with the works of G.D. Birkhoff, dynamical systems in spaces of the billiard type have been intensively studied. The simplest impact law, for absolutely elastic impacts, can be described geometrically as the equality of angles before and after a collision with a boundary of the billiard space. This law, for simplicity, will be assumed in the present paper.

An elementary observation is that the dynamical system of a billiard type with a uniform motion can be modeled by the simple impulsive second-order system

$$\begin{cases} \ddot{x}(t) = 0, & \text{for } t \geq 0, x(t) \in intK \\ \dot{x}(s+) = \dot{x}(s) + I(x(s), \dot{x}(s)), & \text{if } x(s) \in \partial K, \end{cases} \tag{1}$$

where $K = \overline{intK} \subset \mathbb{R}^n$ is a compact subset, and I is an impulse function describing the impact law. It is easy to check that for a unit ball $B(0, 1) \subset \mathbb{R}^n$ and for the equality of the angle of incidence and angle of reflection, one has $I(x(s), \dot{x}(s)) = -2\langle x(s), \dot{x}(s)\rangle x(s)$.

Let us imagine a one-dimensional billiard which is not a straight line but a graph of some differential function $\gamma : [a, b] \to \mathbb{R}$. We can think about some hills and valleys on our simple one-dimensional table.

E-mail address: ggabor@mat.umk.pl.

http://dx.doi.org/10.1016/j.jmaa.2016.03.072
0022-247X/

Assume the gravity directed downstairs. Then, the horizontal component of the acceleration is nonzero. In fact, the motion can be described by a more general equation $\ddot{x}(t) = -k\,\mathrm{grad}\gamma(x(t))$, where k is some constant depending on the gravity.

If we allow an external force depending on time (e.g., a wind), we get even more general equation $\ddot{x}(t) = f(t, x(t))$. One can also easily guess that for tables generated by a nondifferentiable but Lipschitz function γ we obtain a second-order differential inclusion $\ddot{x}(t) \in F(t, x(t))$. This all above motivates to investigate the system

$$\begin{cases} \ddot{x}(t) = f(t, x(t)), & \text{for a.e. } t \geq 0, x(t) \in intK \\ \dot{x}(s+) = \dot{x}(s) + I(x(s), \dot{x}(s)), & \text{if } x(s) \in \partial K. \end{cases} \tag{2}$$

Different kinds of problems in billiards are interesting, e.g., existence of periodic motions and their stability, number of distinct periodic trajectories, etc. (see [6] and references therein). We are interested in the boundary value problem of the Dirichlet type: [(2) and $x(0) = x(T) = 0$] motivated by the research explained below.

In last decades a lot of papers on impulsive boundary value problems have been published. Most of them concerns impulses at fixed moments. In this case one can see direct and clear analogies with the approach and results for problems without impulses. Several difficulties appear when impulses depend on the state variable or both the time and state. One meets the case in e.g. differential population models or mechanics where impulses occurs if some quantities attain a suitable barrier. The papers dealing with state-dependent impulsive problems focus attention mainly on initial or periodic problems, and results on the existence, asymptotic behavior or a stability of solutions. We refer e.g. to [1] (and references therein), where the impulsive periodic problem, also for some kinds of second-order differential equations, is studied.

Unfortunately, the Dirichlet impulsive boundary value problem cannot be brought to the first-order one, and different techniques are needed (comp. some recent papers using a variational approach for problems with fixed impulse times, [8,9,12]). Quite recently in [10] the authors examined the problem

$$\begin{cases} \ddot{x}(t) = f(t, x(t)), & \text{for a.e. } t \in [0, T], \\ \dot{x}(s+) = \dot{x}(s) + I(x(s)), & \text{if } s = g(x(s)), \end{cases} \tag{3}$$

where g is a C^1 function satisfying some additional conditions. They provided a new method to solve the problem. Namely, they successfully transformed the problem to the fixed point problem in an appropriate function space. Note that both the impulse function I and the barrier $Graph(g) = \{(x, s); s = g(x)\}$ are not adequate for billiard problems. Indeed, in billiards the impulse depends also on the velocity before the impact. Moreover, the barrier is not a graph of a function with arguments in a phase space. Note also that in [10] the assumptions insist all trajectories go through the barrier without coming back while in billiards trajectories stay on the same side of the barrier after the impact. The technique presented in [10] does not work for the Dirichlet problem in a billiard space. This problem, as far as the author knows, is still unexplored.

Therefore the aim of the paper is to study the existence and multiplicity of solutions to the Dirichlet impulsive boundary value problem

$$\begin{cases} \ddot{x}(t) = f(t, x(t)), & \text{for a.e. } t \in [0, T], x(t) \in int\,K, \\ \dot{x}(s+) = \dot{x}(s) + I(x(s), \dot{x}(s)), & \text{if } x(s) \in \partial K, \\ x(0) = x(T) = 0, \end{cases} \tag{4}$$

where I describes the impact law of