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Sobolev orthogonal polynomials on the unit ball via outward normal derivatives

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ABSTRACT

We analyse a family of mutually orthogonal polynomials on the unit ball with respect to an inner product which involves the outward normal derivatives on the sphere. Using their representation in terms of spherical harmonics, algebraic and analytic properties will be deduced. First, we deduce explicit connection formulas relating classical multivariate ball polynomials and our family of Sobolev orthogonal polynomials. Then explicit representations for the norms and the kernels will be obtained. Finally, the asymptotic behaviour of the corresponding Christoffel functions is studied.

1. Introduction

The term *Sobolev orthogonal polynomials* usually refers to a family of polynomials which are orthogonal with respect to an inner product which simultaneously involves functions and their derivatives. In the one variable case this kind of orthogonality has been studied during the last 25 years, and it constitutes the main subject of a vast literature (see [9] and the references therein).

Sobolev orthogonal polynomials in several variables have a considerably shorter history. There are very few references on the subject and most of them deal with Sobolev orthogonality on the unit ball $\mathbb{B}^d$ of $\mathbb{R}^d$. Usually, the inner product considered is some modification of the classical inner product on the ball

$$\langle f, g\rangle_\mu = \frac{1}{\omega_\mu}\int_{\mathbb{B}^d} f(x)g(x)W_\mu(x)dx,$$

where $W_\mu(x) = (1-\|x\|^2)^\mu$ on $\mathbb{B}^d$, $\mu > -1$, and ω_μ is a normalizing constant such that $\langle 1, 1\rangle_\mu = 1$.

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One of the first works on this subject was a paper by Y. Xu [14], where the inner product

$$\langle f, g\rangle_I = \frac{\lambda}{\sigma_d}\int_{\mathbb{B}^d} \nabla f(x)\cdot\nabla g(x)dx + \frac{1}{\sigma_d}\int_{\mathbb{S}^{d-1}} f(\xi)g(\xi)d\sigma(\xi), \quad \lambda > 0,$$

was considered. Here, $d\sigma$ denotes the surface measure on the sphere $\mathbb{S}^{d-1}$ and σ_d denotes the surface area. In the same article, the author studied another inner product where the second term on the right hand side was replaced by $f(0)g(0)$. In both cases, the central symmetry of the inner products plays an essential role and using spherical polar coordinates a mutually orthogonal polynomial basis is constructed. The polynomials in this basis are expressed in terms of Jacobi polynomials and spherical harmonics mimicking the standard construction of the classical ball polynomials.

In the present paper, we study orthogonal polynomials with respect to the Sobolev inner product

$$\langle f, g\rangle_\mu^S = \frac{1}{\omega_\mu}\int_{\mathbb{B}^d} f(x)g(x)W_\mu(x)dx + \frac{\lambda}{\sigma_d}\int_{\mathbb{S}^{d-1}} \frac{\partial f}{\partial \mathbf{n}}(\xi)\frac{\partial g}{\partial \mathbf{n}}(\xi)d\sigma(\xi),$$

where $\lambda > 0$ and $\frac{\partial}{\partial \mathbf{n}}$ stands for the outward normal derivative operator.

Using again spherical polar coordinates, we shall construct a sequence of mutually orthogonal polynomials with respect to $\langle\cdot,\cdot\rangle_\mu^S$, which depends on a family of Sobolev orthogonal polynomials of one variable. The latter are usually called a *non-diagonal Jacobi Sobolev-type* family of orthogonal polynomials and can be expressed in terms of Jacobi polynomials (see [4]).

Standard techniques provide us explicit connection formulas relating classical multivariate ball polynomials and our family of Sobolev orthogonal polynomials. The explicit representations for the norms and the kernels will be obtained.

A very interesting problem in the theory of multivariate orthogonal polynomials is that of finding asymptotic estimates for the Christoffel functions, because these estimates are related to the convergence of the Fourier series. Asymptotics for Christoffel functions associated to the classical orthogonal polynomials on the ball were obtained by Y. Xu in 1996 (see [13]). Recently, more general results on the asymptotic behaviour of the Christoffel functions were established by Kroó and Lubinsky [7,8]. Those results include estimates in a quite general case where the orthogonality measure satisfies some regularity conditions.

Since our orthogonal polynomials do not fit into the above mentioned case, the asymptotic of the Christoffel functions deserves special attention. Not surprisingly, our results show that in any compact subset of the interior of the unit ball Christoffel functions in the Sobolev case behave exactly as in the classical case, see Theorem 4. On the sphere the situation is quite different and we can perceive the influence of the outward normal derivatives in the inner product, see Theorem 3.

The paper is organized as follows. In the next section, we state the background materials on orthogonal polynomials on the unit ball and spherical harmonics that we will need later. In Section 3, using spherical polar coordinates we construct explicitly a sequence of mutually orthogonal polynomials with respect to $\langle\cdot,\cdot\rangle_\mu^S$. Those polynomials are given in terms of spherical harmonics and a family of univariate Sobolev orthogonal polynomials in the radial part, their properties are studied in Section 4. In Section 5, we deduce explicit connection formulas relating classical multivariate ball polynomials and our family of Sobolev orthogonal polynomials. Moreover, an explicit representation for the kernels is obtained. The asymptotic behaviour of the corresponding Christoffel functions is studied in Section 6. And finally, in Section 7, we consider the special case $d = 2$.