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Dimension bounds for invariant measures of bi-Lipschitz iterated function systems

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ABSTRACT

We study probabilistic iterated function systems (IFSs), consisting of a finite or infinite number of average-contracting bi-Lipschitz maps on $\mathbb{R}^d$. If our strong open set condition is also satisfied, we show that both upper and lower bounds for the Hausdorff and packing dimensions of the invariant measure can be found. Both bounds take on the familiar form of ratio of entropy to the Lyapunov exponent. Proving these bounds in this setting requires methods which are quite different from the standard methods used for average-contracting IFSs.

1. Introduction

When studying the dimension of measures for probabilistic iterated function systems, there are two common approaches. One method is to examine small balls centered at typical points for the Markov chain, and estimate the logarithmic densities (2.7)–(2.10). Since this method requires some intimate understanding of the geometry of the system, it has proven to be effective primarily in the simple case of similitudes satisfying the so-called open set condition, which limits the overlap of the maps. In this scenario, the exact dimensional value of the invariant measure μ has been determined in e.g. [1] (the finite case) and later generalized in [6] (the infinite case).

Another common approach is to assume very little of the maps and only search for an upper bound s for the dimension of μ, by explicitly constructing a set of full μ-measure whose s-dimensional Hausdorff measure is zero. Usually only average contractivity (a notion introduced in [3]) is assumed. This approach has been used in e.g. [9,5,4]. While this method is easier to apply to a wider class of maps, the drawbacks are that it does not give any lower bound for the dimension, nor does it shed any light on the packing dimension of μ.

The aim of this paper is to obtain results for a special class of IFS by merging these two approaches in a way, primarily by extending techniques used for analyzing strictly contracting IFSs. We will focus on the

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case where the maps are bi-Lipschitz. By assuming average contractivity, the support of μ is not necessarily bounded and many of the initial assumptions in the case of similitudes fail. We will extend some ideas from the second approach to show that we can still find lower and upper bounds for both the Hausdorff and packing dimensions of μ.

The motivation for this paper rises from the fact that lower bounds of the dimension of μ are not commonly investigated. One example is [8], where μ however is required to have bounded support. Here we will prove the intuitive result that the lower Lipschitz condition implies a lower bound in a similar way that the upper Lipschitz condition usually implies an upper bound. Our particular scenario is interesting because the method we use to prove the dimension bounds (the lower one in particular) differs from the "standard" technique for similar settings.

We will define a (probabilistic) iterated function system (IFS) as a set $\mathbb{X} \subset \mathbb{R}^d$ associated with a family of maps $\mathscr{W} = \{w_i\}_{i\in M}$, $w_i : \mathbb{X} \to \mathbb{X}$, where the maps are chosen independently according to a probability vector $\mathbf{p} = \{p_i\}_{i\in M}$, where $p_i > 0$ for all $i \in M$ and $\sum_{i\in M} p_i = 1$. The index set M is either finite or (countably) infinite. We will assume that the maps are bi-Lipschitz: for every $i \in M$ there exist constants Γ_i and γ_i, such that $0 < \gamma_i < \Gamma_i$ and

$$|w_i(x) - w_i(y)| \geq \gamma_i |x - y| \tag{1.1}$$

$$|w_i(x) - w_i(y)| \leq \Gamma_i |x - y|, \tag{1.2}$$

for all $x, y \in \mathbb{X}$. We will use the notation $\{\mathbb{X}, \mathscr{W}, \mathbf{p}\}$ for an IFS as described above.

Let $M^\infty = M \times M \times \ldots$ and define the infinite-fold product probability measure on M^∞ as $\mathbb{P} = \mathbf{p} \times \mathbf{p} \times \ldots$. We define the mapping $\pi : M^\infty \to \mathbb{R}^d$ by

$$\pi(i_1, i_2, \ldots) = \lim_{n\to\infty} w_{i_1} \circ \cdots \circ w_{i_n}(x_0) \tag{1.3}$$

if the limit exists.

For an IFS satisfying (1.2), it is well known (see [3]) that if there exists some $x \in \mathbb{X}$ such that the conditions

$$\text{(i)} \quad \sum_{i\in M} p_i \Gamma_i < \infty \tag{1.4}$$

$$\text{(ii)} \quad \sum_{i\in M} p_i |w_i(x) - x| < \infty \tag{1.5}$$

$$\text{(iii)} \quad -\infty < \sum_{i\in M} p_i \log \Gamma_i < 0 \tag{1.6}$$

hold, then there exists a unique probability measure μ on $\mathbb{X}$ such that

$$\mu = \mathbb{P} \circ \pi^{-1}$$

on $\mathscr{B}(\mathbb{X})$, the Borel algebra on $\mathbb{X}$. Alternatively, μ may be defined as the unique measure satisfying

$$\mu = \sum_{i\in M} p_i \mu \circ w_i^{-1}.$$

The operator $\sum_{i\in M} p_i \mu \circ w_i^{-1}$ is sometimes called the Markov operator, and μ is called the invariant measure of the IFS. An IFS satisfying conditions (1.4)–(1.6) is said to satisfy average contractivity. In this case there may be expanding maps in $\mathscr{W}$ and thus π does not necessarily exist for all $\mathbf{i} \in M^\infty$. However, the limit (1.3) exists $\mathbb{P}$-a.e. and does not depend on x_0.