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The Sarason sub-symbol and the recovery of the symbol of densely defined Toeplitz operators over the Hardy space

Joel A. Rosenfeld [1]

Nonlinear Controls and Robotics (NCR) Laboratory, Department of Mechanical and Aerospace Engineering, University of Florida, Gainesville, FL 32611, United States

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ABSTRACT

While the symbol map for the collection of bounded Toeplitz operators is well studied, there has been little work on a symbol map for densely defined Toeplitz operators. In this work a family of candidate symbols, the Sarason sub-symbols, is introduced as a means of reproducing the symbol of a densely defined Toeplitz operator. This leads to a partial answer to a question posed by Donald Sarason in 2008. In the bounded case the Toeplitzness of an operator can be classified in terms of its Sarason sub-symbols. This justifies the investigation into the application of the Sarason sub-symbols on densely defined operators. It is shown that analytic closed densely defined Toeplitz operators are completely determined by their Sarason sub-symbols, and it is shown for a broader class of operators that they extend closed densely defined Toeplitz operators (of multiplication type).

1. Introduction

The study of bounded Toeplitz operators over the Hardy space $H^2(\mathbb{T})$ is a well developed subject where there are several equivalent definitions of a Toeplitz operator. The simplest definition of a bounded Toeplitz operator is an extension of the definition of a Toeplitz matrix. In this case an operator, T, is called a Toeplitz operator if the matrix representation of the operator, with respect to the orthonormal basis $\{e^{in\theta}\}_{n=0}^{\infty}$ is constant along the diagonals. Algebraically, this can be represented as $S^*TS = T$. Here $S = M_z$ is the shift operator for the Hardy space. If the coefficients corresponding to each diagonal of the matrix are the Fourier coefficients of a function $\phi \in L^\infty(\mathbb{T})$, then $T = PM_\phi$. Here P is the projection from $L^2(\mathbb{T}) \to H^2(\mathbb{T})$, and M_ϕ is the bounded multiplication operator from $H^2(\mathbb{T}) \to L^2(\mathbb{T})$ given by $M_\phi f = \phi f$. Finally the converse is true, the bounded operator given by $T_\phi = PM_\phi$ with $\phi \in L^\infty(\mathbb{T})$ satisfies $S^*TS = T$.

When the bounded condition is relaxed to closed and densely defined, the corresponding definitions of Toeplitz operators are no longer equivalent. For instance, if the coefficients of an upper triangular matrix,

E-mail address: joelar@ufl.edu.

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$$\begin{pmatrix} a_1 & a_2 & a_3 & \\ 0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_1 & \\ & \vdots & & \ddots \end{pmatrix},$$

are the coefficients of a Smirnov class function, $\phi \in N^+$, then the operator defined by the closure of this matrix, call it T, is densely defined, and the operator is the adjoint of a densely defined multiplication operator (an analytic Toeplitz operator) M_ϕ. Morally M_ϕ is given by a lower triangular matrix of the form

$$\begin{pmatrix} \bar{a}_1 & 0 & 0 & \\ \bar{a}_2 & \bar{a}_1 & 0 & \cdots \\ \bar{a}_3 & \bar{a}_2 & \bar{a}_1 & \\ & \vdots & & \ddots \end{pmatrix}$$

though $\langle M_\phi z^n, z^m \rangle$ may not be well-defined, since z^n isn't necessarily in the domain of M_ϕ. Unlike its bounded counterpart, T cannot be represented by a multiplication operator as $PM_{\bar{\phi}}$ since its domain is strictly larger than the domain of $M_{\bar{\phi}}$. The operator T does satisfy the following algebraic equations:

1. $D(T)$ is S-invariant,
2. $S^*TS = T$, and
3. If $f \in D(T)$ and $f(0) = 0$, then $S^*f \in D(T)$.

These can be seen as the densely defined analogue of the algebraic condition for bounded Toeplitz operators. Therefore T satisfies the algebraic conditions for being a Toeplitz operator, but is not a Toeplitz operator in the multiplication sense. However, T is a closed extension of a multiplication type Toeptliz operator. At the close of [28] the following problem was posed:

Question 1. Is it possible to characterize those closed densely defined operators T on $H^2(\mathbb{T})$ with the above three properties? Moreover, is every closed densely defined operator on $H^2(\mathbb{T})$ that satisfies these conditions determined in some sense by a symbol?

This paper aims to address the second half of this question. If a closed densely defined operator, T, satisfies the three algebraic conditions above, henceforth a *Sarason–Toeptliz* operator, then is T the extension of an operator of the form PM_ϕ where M_ϕ is a densely defined multiplication operator from H^2 to L^2?

Various investigations into unbounded operators have been performed as far back as the 1950s. Hartman and Wintner [20] investigated the self-adjointness of unbounded Toeplitz matrices, and Hartman continued the work in [19]. Rosenblum in [25] and Rovnyak in [26] investigated the resolvent of unbounded Toeplitz operators. Further results for unbounded analytic and co-analytic Toeplitz operators can be found in [21,28–30].

For bounded Toeplitz operators the recovery of the symbol of a Toeplitz operator can be achieved through the symbol map on $\mathcal{T}$, the algebra generated by the collection of Toeplitz operators in $\mathcal{L}(H^2)$. Douglas demonstrated that there is a unique multiplicative mapping, ϕ from $\mathcal{T}$ to L^∞ such that $\phi(T_f T_g) = \phi(T_f)\phi(T_g) = fg$ [1,12]. This fact was proven again in [5] by Halmos and Barria using the limits along the diagonals of a Toeplitz matrix in order to find the symbol in L^∞.

The Hardy space can be identified with analytic functions of the disc $\mathbb{D}$ such that the Taylor coefficients of these functions are square summable. By this viewpoint, H^2 is a reproducing kernel Hilbert space (RKHS) over $\mathbb{D}$ with the kernel functions $k_w(z) = (1 - \bar{w}z)^{-1}$ for $|w| < 1$.

In the case of bounded Toeplitz operators, the Berezin transform, a tool particular to the study of RKHSs, is sufficient for the recovery of the of L^∞ functions via radial limits of the Berezin transform of a