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Polynomials with rational generating functions and real zeros

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ABSTRACT

This paper investigates the location of the zeros of a sequence of polynomials generated by a rational function with a binomial-type denominator. We show that every member of a two-parameter family consisting of such generating functions gives rise to a sequence of polynomials $\{P_m(z)\}_{m=0}^{\infty}$ that is eventually hyperbolic. Moreover, the real zeros of the polynomials $P_m(z)$ form a dense subset of an interval $I \subset \mathbb{R}^+$, whose length depends on the particular values of the parameters in the generating function.

1. Introduction

The study of the location of zeros of polynomials is one of the oldest endeavors in mathematics. The prolific mathematical production of the nineteenth century included a number of advances in this endeavor. The unsolvability of the general quintic equation together with the fundamental theorem of algebra led to the consideration that when it comes to extracting information about the zeros of a complex polynomial from its coefficients, one should perhaps strive to determine subsets of $\mathbb{C}$ where the zeros must lie,[1] rather than looking for the exact location of the zeros. The development of the Cauchy theory for analytic functions provided some of the classic machinery suitable for such investigations, including Rouché's theorem, the argument principle and the Routh–Hurwitz condition for left-half plane stability. We mention these results not only because they are powerful tools, but also because they embody a fundamental dichotomy. Explicit criteria for the location of the zeros of a polynomial in terms of its coefficients may severely restrict the domain to which they apply, whereas ubiquitous applicability of a theorem to various domains may render the result difficult to use.

The Gauss–Lucas theorem, relating the location of the zeros of $p'(z)$ to those of the polynomial $p(z)$, pioneered a new approach to an old question: instead of studying the zeros of a function, one can study the

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[1] Although the publication of *La Géometrie* predates the early XIXth century by almost two hundred years, from this perspective, Descartes' rule of signs should be mentioned, as it gives information on the number of real positive, real negative, and non-real zeros of a real polynomial.

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behavior of the zero set of a function under certain operators. In this light, given the Taylor expansion of a (real) entire function $f(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$, one can interpret $f(x)$ as the result of the sequence $\{\gamma_k\}_{k=0}^{\infty}$ acting on the function e^x by forming a Hadamard product.[2] Thus, complex sequences have a dual nature: they are coefficients of 'polynomials of infinite degree' (à la Euler), as well as linear operators on $\mathbb{C}[x]$. G. Pólya and J. Schur's 1914 paper [10] was a major milestone in understanding how sequences (as linear operators) affect the location of the zeros of polynomials. More precisely, Pólya and Schur gave a classification of real sequences that preserve reality of zeros of real polynomials, and initiated a research program on stability preserving linear operators on circular domains, which was recently completed by J. Borcea and P. Brändén [2].

Since the work of Pólya and Schur, the study of sequences as linear operators on $\mathbb{R}[x]$ has attracted a great amount of attention. We remark here only that real sequences, when looked at as operators on $\mathbb{R}[x]$, admit a representation as a formal power series

$$\{\gamma_k\}_{k=0}^{\infty} \sim \sum_{k=0}^{\infty} Q_k(x) D^k,$$

where D denotes differentiation, and the $Q_k(x)$s are polynomials with degree k or less (see for example [9]). In [5] the first author and A. Piotrowski study the extent to which the 'generated' sequence $\{Q_k(x)\}_{k=0}^{\infty}$ encodes the reality preserving properties of the sequence $\{\gamma_k\}_{k=0}^{\infty}$, and find that if this latter sequence is a Hermite-diagonal[3] reality preserving operator, then all of the $Q_k(x)$s must have only real zeros.

The present paper extends the works of S. Beraha, J. Kahane, and N.J. Weiss [1], A. Sokal [11] and K. Tran [12,13], by studying a large family of generating functions which give rise to sequences of polynomials with only real zeros. Despite the many similarities between our work and that of R. Boyer and W. Goh [3,4], there is a key difference in the nature of our results. While Boyer and Goh study zero attractors, we study the zero loci of sequences of polynomials. Our main result (see Theorem 1) concerns a sequence of polynomials, whose generating function is rational with a binomial-type denominator.

Theorem 1. *Let $n, r \in \mathbb{N}$ such that $\max\{r, n\} > 1$, and set $D_{n,r}(t, z) := (1-t)^n + zt^r$. For all large m, the zeros of the polynomial $P_m(z)$ generated by the relation*

$$\sum_{m=0}^{\infty} P_m(z) t^m = \frac{1}{D_{n,r}(t,z)} \tag{1}$$

lie on the interval

$$I = \begin{cases} (0, \infty) & \text{if } n, r \geq 2 \\ (0, n^n/(n-1)^{n-1}) & \text{if } r = 1 \\ ((r-1)^{r-1}/r^r, \infty) & \text{if } n = 1 \end{cases}$$

Furthermore, if $\mathcal{Z}(P_m)$ denotes the set of zeros of the polynomial $P_m(z)$, then $\bigcup_{m \gg 1} \mathcal{Z}(P_m)$ is dense in I.

Although this result is asymptotic in nature, we do believe that in fact all of the generated polynomials have only real zeros. Given that we have no proof of this claim at this time, we pose this stronger statement as an open problem (see Problem 12 in Section 4).

[2] We direct the reader to the beautiful works of Hardy [7] and Ostrovskii [8] concerning the zero loci of certain entire functions obtained this way.

[3] By $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ being a Hermite diagonal operator we simply mean that $\Gamma[H_n(x)] = \gamma_n H_n(x)$ for all n, where $H_n(x)$ denotes the nth Hermite polynomial.