J. Math. Anal. Appl. 443 (2016) 707–731

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Mellin convolution operators in Bessel potential spaces ☆

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ARTICLE INFO

Article history:
Received 31 March 2015
Available online 27 May 2016
Submitted by W.L. Wendland

Keywords:
Fourier convolutions
Mellin convolutions
Bessel potential spaces
Operator algebras
Symbol
Fixed singularities

ABSTRACT

Mellin convolution equations acting in Bessel potential spaces are considered. The study is based upon two results. The first one concerns the interaction of Mellin convolutions and Bessel potential operators (BPOs). In contrast to the Fourier convolutions, BPOs and Mellin convolutions do not commute and we derive an explicit formula for the corresponding commutator in the case of Mellin convolutions with meromorphic symbols. These results are used in the lifting of the Mellin convolution operators acting on Bessel potential spaces up to operators on Lebesgue spaces. The operators arising belong to an algebra generated by Mellin and Fourier convolutions acting on $\mathbb{L}_p$-spaces. Fredholm conditions and index formulae for such operators have been obtained earlier by one of the authors and are employed here. The results of the present work have numerous applications in boundary value problems for partial differential equations, in particular, for equations in domains with angular points.

0. Introduction

Boundary value problems for elliptic equations in domains with angular points play an important role in applications and have a rich and exciting history. A prominent representative of this family is the Helmholtz equation. In the classical $\mathbb{W}^1$-setting, the existence and uniqueness of the solution of coercive systems with various types of boundary conditions and various elliptic and even non-linear partial differential operators are easily obtainable by using the celebrated Lax–Milgram Theorem (see, e.g., [8,30] and the recent paper [21] where Laplace–Beltrami equations are considered on smooth surface with Lipschitz boundary). Similar problems arise in new applications in physics, mechanics and engineering. Thus recent publications on nano-photonics [1,25] deal with physical and engineering problems described by BVPs for the

☆ This work was carried out when the second author visited the Universiti Brunei Darussalam (UBD). The support of UBD provided via Grant UBD/GSR/S&T/19 is highly appreciated. The work is also supported by the Georgian National Science Foundation, Contracts No. 13/14 and No. 31/39.

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http://dx.doi.org/10.1016/j.jmaa.2016.05.043
0022-247X/

Helmholts equation in $2D$ domains with angular points. They are investigated with the help of a modified Lax–Milgram Lemma for so-called T-coercive operators. Similar problems occur for the Lamé systems in elasticity, Cauchy–Riemann systems, Carleman–Vekua systems in generalized analytic function theory etc.

Despite an impressive number of publications and ever growing interest to such problems, the results available to date are not complete. In particular, serious difficulties arise if information on the solvability in non-classical setting in the Sobolev spaces $\mathbb{W}^1_p$, $1 < p < \infty$ is required, and one wants to study the solvability of equivalent boundary integral equations in the trace spaces $\mathbb{W}^{1-1/p}_p$ on the boundary. Integral equations arising in this case often have fixed singularities in the kernel and are of Mellin convolution type. For example, [6] describes how model BVPs in corners emerge from the localization of BVP for the Helmholtz equation in domains with Lipschitz boundary. Consequently, an attempt to study the corresponding Mellin convolution operators in Bessel potential spaces has been undertaken in [19]. However, the main Theorem 2.7 and Theorem 4.1 (based on Theorem 2.7) are incorrect. The aim of the present work is to provide correct formulations and proofs of Theorem 2.7 and 4.1 from [19]. We also hope that the results of the present paper will be helpful in further studies of boundary value problems for various elliptic equations in Lipschitz domains.

Consider the following BVP with the mixed Dirichlet–Neumann boundary conditions

$$\begin{cases} \Delta u(x) + k^2 u(x) = 0, & x \in \Omega_\alpha, \\ u^+(t) = g(t), & t \in \mathbb{R}^+, \\ (\partial_{\boldsymbol{\nu}} u)^+(t) = h(t), & t \in \mathbb{R}_\alpha \end{cases} \tag{1}$$

in the corner Ω_α of magnitude α,

$$\partial\Omega_\alpha = \mathbb{R}^+ \cup \mathbb{R}_\alpha, \quad \mathbb{R}^+ = (0, \infty),$$
$$\mathbb{R}_\alpha := \{te^{i\alpha} = (t\cos\alpha, t\sin\alpha) \ : \ t \in \mathbb{R}^+\}$$

with a complex wave number $\operatorname{Im} k \neq 0$. In [20] the BVP (1) is reduced to the following equivalent system of boundary integral equations on $\mathbb{R}^+$:

$$\begin{cases} \varphi + \dfrac{1}{2}\left[\boldsymbol{K}^1_{e^{i\alpha}} + \boldsymbol{K}^1_{e^{-i\alpha}}\right]\psi = G_1, \\ \psi - \dfrac{1}{2}\left[\boldsymbol{K}^1_{e^{i\alpha}} + \boldsymbol{K}^1_{e^{-i\alpha}}\right]\varphi = H_1. \end{cases} \tag{2}$$

Here

$$\boldsymbol{K}^1_{e^{\pm i\alpha}}\psi(t) := \frac{1}{\pi}\int_0^\infty \frac{\psi(\tau)d\tau}{t - e^{\pm i\alpha}\tau}, \qquad 0 < |\alpha| < \pi, \tag{3}$$

are Mellin convolution operators with homogeneous kernels of order -1 (see e.g. [16,17] and Section 1 below), also called integral equations with fixed singularities in the kernel. Similar integral operators arise in the theory of singular integral equations with complex conjugation if the contour of integration possesses corner points. A complete theory of such equations was worked out by R. Duduchava and T. Latsabidze, whereas various approximation methods have been investigated in [13]. For a more detailed survey of this theory, applications in elasticity, and numerical methods for the corresponding equations we refer the reader to [16,17,32] and [11,12]. Note that a similar approach has been employed by M. Costabel and E. Stephan [9,10] in order to study boundary integral equations on curves with corner points.