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Segal–Wilson approach to integrable systems and Riemann–Hilbert problems

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ABSTRACT

In this paper a theory is developed for obtaining families of solutions to the KdV equation by formulating a Riemann–Hilbert problem with an appropriate shift. The theory builds on the classical work of Segal and Wilson [17] in which families of solutions are indexed on closed subspaces W of a space of functions on the unit circle admitting a direct sum decomposition $H = H^+ \oplus H^-$ (H^+, H^- are subspaces of functions holomorphic respectively inside and outside the unit disk). The theory developed in this paper lends itself easily to obtaining explicit solutions. Examples where the subspace W can be associated to soliton type solutions are considered. More complex systems where singularities and Riemann surfaces play a role are also presented. In the last section the connection of our results to the τ-function is analyzed. The theory developed in this paper can easily be applied to other integrable systems and, eventually, to discrete integrable systems.

1. Introduction

As is well known the KdV equation,

$$-4\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x}, \tag{1.1}$$

is the compatibility condition between the equations (see *e.g.* [13, Ch. 3])

$$\left(\frac{\partial^2}{\partial x^2} + u\right)\phi = \lambda^2\phi \tag{1.2}$$

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$$\frac{\partial \phi}{\partial t} = P_u \phi, \tag{1.3}$$

where P_u is the operator given by

$$P_u \phi = \left(4 \frac{\partial^3}{\partial x^3} - 6u \frac{\partial}{\partial x} - 3u' \right) \phi,$$

with $u' := \frac{\partial u}{\partial x}$. In (1.2) and (1.3) λ is a spectral parameter. The function ϕ is usually called the Baker–Akhiezer function for the problem (1.2), (1.3).

In this paper we concentrate on obtaining families of solutions to the Schrödinger equation (or the KdV equation) indexed on function loops defined on the unit circle (called *symbols* below).

Our approach builds on the fundamental results of Segal and Wilson ([17] and [13]), which in turn may be seen as parallel to the so-called algebraic–geometric method of Kričever, Dubrovin and others ([14] and [8]). This approach is quite distinct from the true inverse scattering methods that aim at solving the *initial–boundary value problem* (Cauchy problem) for the KdV equation and other integrable partial differential equations ([1,5,18,9]). A difficulty inherent to the Cauchy problem approach lies in the fact that the associated Riemann–Hilbert problems are very hard to solve and almost impossible to yield explicit solutions except for the reflectionless cases (soliton and multi-soliton solutions) – see, for example, [1, Chapter 2, Eq. 2.3.7].

In fact in [1] it is shown that the solution of the associated Riemann–Hilbert problem involves the factorization on the real line of a symbol G of the form

$$G(\lambda) = \begin{bmatrix} 1 + \rho(\lambda)\overline{\rho}(\lambda) & \rho(\lambda)e^{2i\lambda x} \\ \overline{\rho}(\lambda)e^{-2i\lambda x} & 1 \end{bmatrix}, \qquad (\lambda \in \mathbb{R}), \tag{1.4}$$

where ρ is the reflection coefficient. The factorization of G is very difficult to study except for $\rho(\lambda) = 0$ or ρ rational (soliton solutions) as the exponential $\exp(2i\lambda x)$ is an inner function on the upper half plane. To the authors' knowledge no theory exists for the analysis of this problem. At this point it should be clear that there exist several approaches to associate a Riemann–Hilbert problem to the inverse-scattering study (*cf.* [1,18,10] and [11]) but all lead to a Riemann–Hilbert problem involving a symbol of the form (1.4).

Our approach, derived from the theory of Segal and Wilson, involves solving a Riemann–Hilbert on the unit circle S^1 with a symbol of the form

$$G(\lambda) = \begin{bmatrix} \gamma_a & \gamma_b e^{-2\lambda x} \\ \gamma_b(\text{-})e^{2\lambda x} & \gamma_a(\text{-}) \end{bmatrix} \qquad (\lambda \in S^1), \tag{1.5}$$

where γ_a and γ_b are scalar symbols on S^1 and, for $f\colon S^1 \to \mathbb{C}$, $f(\text{-})(\lambda) := f(-\lambda)$. In (1.5) the exponentials $\exp(\pm 2\lambda x)$ are outer functions relative to the unit disk [7], which makes it easier to analyze the factorization of G.

However to the best of the authors' knowledge the approach of Segal and Wilson has not been used to yield explicit solutions except again for the soliton and multi-solitons cases. The work presented in this paper gives a general method to obtain solutions to the KdV equation (extendible to other integrable partial differential equations) by expressing them as solutions to a class of Riemann–Hilbert problems with shift indexed on pairs of symbols defined on the unit circle. This, in turn, involves the factorization of a symbol of the form (1.5).

To explain briefly our approach let H be a Banach space of complex-valued functions on the unit circle, admitting a direct sum decomposition $H = H^+ \oplus H^-$. In [17] and [13] it is shown that with each closed subspace $W \subset H$ satisfying $\lambda^2 W \subset W$ we can associate a unique Baker–Akhiezer function