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The power series method for nonlocal and nonlinear evolution equations

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ABSTRACT

The initial value problem for a 4-parameter family of nonlocal and nonlinear evolution equations with data in a space of analytic functions is solved by using a power series method in abstract Banach spaces. In addition to determining the power series expansion of the solution, this method also provides an estimate of the analytic lifespan expressed in terms of the norm of the initial data, thus establishing an abstract Cauchy–Kovalevsky type theorem for these equations.

1. Introduction and results

In this work we prove an abstract Cauchy–Kovalevsky theorem for the following 4-parameter family of Camassa–Holm type equations

$$\begin{aligned} u_t + u^k u_x - a u^{k-2} u_x^3 + \partial_x (1-\partial_x^2)^{-1} \Big[\frac{b}{k+1}\, u^{k+1} + c\, u^{k-1} u_x^2 - a\,(k-2)\, u^{k-3} u_x^4 \Big] \\ + (1-\partial_x^2)^{-1} \Big[\big[k\,(k+2) - 8a - b - c\,(k+1) \big] u^{k-2} u_x^3 - 3a\,(k-2)\, u^{k-3} u_x^3\, u_{xx} \Big] = 0, \end{aligned} \tag{1.1}$$

which was introduced in [28] and is referred there as the k-abc-equation. The three parameters a, b and c range over the real numbers while k is a positive integer, whose value depends on a. If $a \neq 0$ then $k \geq 2$ and the presence of the term $au^{k-2}u_x^3$ makes k-abc-equation a nonlocal and nonlinear equation which is not

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quasilinear. For $k = 2$ and $c = (6 - 6a - b)/2$, we obtain the ab-family of equations (ab-equation) with cubic nonlinearities

$$u_t + u^2 u_x - a u_x^3 + \partial_x (1 - \partial_x^2)^{-1} \Big[\frac{b}{3} u^3 + \frac{6 - 6a - b}{2} u u_x^2 \Big] + (1 - \partial_x^2)^{-1} \Big[\frac{2a + b - 2}{2} u_x^3 \Big] = 0, \tag{1.2}$$

which was also introduced in [28] and which contains two well-known integrable equations with cubic nonlinearities. In fact, for $a = 1/3$ and $b = 2$ the ab-equation gives the Fokas–Olver–Rosenau–Qiao (FORQ) equation (also known as the modified Camassa–Holm equation)

$$\partial_t u + u^2 \partial_x u - \tfrac{1}{3} (\partial_x u)^3 + \partial_x (1 - \partial_x^2)^{-1} \Big[\frac{2}{3} u^3 + u \, (\partial_x u)^2 \Big] + (1 - \partial_x^2)^{-1} \Big[\frac{1}{3} \, (\partial_x u)^3 \Big] = 0, \tag{1.3}$$

which was derived in different ways by Fokas [18], Olver and Rosenau [46] and Qiao [49], and also appeared in a work by Fuchssteiner [19]. For $a = 0$ and $b = 3$ the ab-equation gives the Novikov equation (NE)

$$u_t + u^2 u_x + \partial_x (1 - \partial_x^2)^{-1} \Big[u^3 + \frac{3}{2} u u_x^2 \Big] + (1 - \partial_x^2)^{-1} \Big[\frac{1}{2} u_x^3 \Big] = 0, \tag{1.4}$$

which was derived by V. Novikov in [44], where he provides a classification of all integrable CH-type equations with quadratic and cubic nonlinearities.

Finally, for $a = 0$ and $c = (3k - b)/2$ the k-abc-equation makes sense for all $k \geq 1$ and gives the following generalized Camassa–Holm equation (g-kbCH)

$$u_t + u^k u_x + (1 - \partial_x^2)^{-1} \partial_x \Big[\frac{b}{k+1} u^{k+1} + \frac{3k - b}{2} u^{k-1} u_x^2 \Big] + (1 - \partial_x^2)^{-1} \Big[\frac{(k-1)(b-k)}{2} u^{k-2} u_x^3 \Big] = 0, \tag{1.5}$$

which is a quasilinear equation with $(k + 1)$ order nonlinearities and which was studied in [25] and [21]. When $k = 1$ the g-kbCH equation gives the well-known b-equation

$$\underbrace{m_t}_{\text{evolution}} + \underbrace{u m_x}_{\text{convection}} + \underbrace{b u_x m}_{\text{stretching}} = 0, \quad m = u - u_{xx}, \tag{1.6}$$

having quadratic nonlinearities. In this local form it was introduced by Holm and Staley [32,33] and it expresses a balance between evolution, convection and stretching. The b-equation (1.6) contains two integrable members, namely the Camassa–Holm (CH) equation that corresponds to $b = 2$ and the Degasperis–Procesi (DP) equation that corresponds to $b = 3$. Mikhailov and Novikov [41] proved that there are no other integrable members of the b-equation. Furthermore, V. Novikov [45] recently proved that the only other integrable member of the g-kbCH equation (1.5) apart from CH and DP is the NE (1.4). To summarize, thus far it is known that the k-abc-equation (1.1) contains four integrable equations, CH, DP, FORQ and NE. However, the existence of other integrable members of the k-abc-equation remains an open question. In fact, integrability theory provides one of the motivations for studying such nonlocal equations.

Another motivation for studying equations like k-abc-equation is the quest for equations capturing wave breaking and peaking, which goes back to Whitham, who articulates it in his 1974 book [55] (p. 477) as follows: *"Although both breaking and peaking, as well as criteria for the occurrence of each, are without doubt contained in the equations of the exact potential theory, it is intriguing to know what kind of simpler mathematical equation could include all these phenomena."* It is remarkable that the k-abc-equation has peakon traveling wave solutions for all values of the four parameters k, a, b and c. These, including multipeakons, have been derived in [28]. The peakon solutions in the non-periodic case can be written in the following form

$$u(x,t) = \gamma e^{-\left| x - (1-a)\gamma^k t \right|}, \quad \gamma \in \mathbb{R}. \tag{1.7}$$