J. Math. Anal. Appl. 443 (2016) 848–867

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Oscillation and variation for semigroups associated with Bessel operators

Huoxiong Wu [a], Dongyong Yang [a,*], Jing Zhang [a,b]

[a] School of Mathematical Sciences, Xiamen University, Xiamen 361005, China
[b] School of Mathematics and Statistics, Yili Normal College, Yining, Xinjiang 835000, China

ARTICLE INFO

Article history:
Received 25 January 2016
Available online 30 May 2016
Submitted by J. Xiao

Keywords:
Oscillation
Variation
Bessel operator
Poisson semigroup
Heat semigroup

ABSTRACT

Let $\lambda > 0$ and $\triangle_\lambda := -\frac{d^2}{dx^2} - \frac{2\lambda}{x}\frac{d}{dx}$ be the Bessel operator on $\mathbb{R}_+ := (0, \infty)$. The authors show that the oscillation operator $\mathcal{O}(P_*^{[\lambda]})$ and variation operator $\mathcal{V}_\rho(P_*^{[\lambda]})$ of the Poisson semigroup $\{P_t^{[\lambda]}\}_{t>0}$ associated with Δ_λ are both bounded on $L^p(\mathbb{R}_+, dm_\lambda)$ for $p \in (1, \infty)$, $\mathrm{BMO}(\mathbb{R}_+, dm_\lambda)$, from $L^1(\mathbb{R}_+, dm_\lambda)$ to $L^{1,\infty}(\mathbb{R}_+, dm_\lambda)$, and from $H^1(\mathbb{R}_+, dm_\lambda)$ to $L^1(\mathbb{R}_+, dm_\lambda)$, where $\rho \in (2, \infty)$ and $dm_\lambda(x) := x^{2\lambda}\,dx$. As an application, an equivalent characterization of $H^1(\mathbb{R}_+, dm_\lambda)$ in terms of $\mathcal{V}_\rho(P_*^{[\lambda]})$ is also established. All these results hold if $\{P_t^{[\lambda]}\}_{t>0}$ is replaced by the heat semigroup $\{W_t^{[\lambda]}\}_{t>0}$.

1. Introduction and statement of main results

Let $(\mathcal{X}, \mu)$ be a measure space and $\mathcal{T}_* := \{T_\epsilon\}_{\epsilon>0}$ a family of operators bounded on $L^p(\mathcal{X}, \mu)$ for $p \in (1, \infty)$ such that $\lim_{\epsilon\to 0} T_\epsilon f$ exists in some sense. A classical way to measure the speed of convergence of $\{T_\epsilon\}_{\epsilon>0}$ is to study square functions of the type $(\sum_{i=1}^\infty |T_{\epsilon_i} f - T_{\epsilon_{i+1}} f|^2)^{1/2}$, where $\epsilon_i \to 0$. Recently, other expressions have been considered, among which are the ρ-variation and the oscillation operators; see, for instance, [7–11,14,15,17,18,20]. Recall that variation operator $\mathcal{V}_\rho(\mathcal{T}_* f)$ is defined by

$$\mathcal{V}_\rho(\mathcal{T}_* f)(x) := \sup_{\epsilon_i \searrow 0} \left(\sum_{i=1}^\infty |T_{\epsilon_{i+1}} f(x) - T_{\epsilon_i} f(x)|^\rho \right)^{1/\rho}, \tag{1.1}$$

where the supremum is taken over all sequences $\{\epsilon_i\}$ decreasing to zero. The oscillation operator $\mathcal{O}(\mathcal{T}_* f)$ can be introduced as

* Corresponding author.
E-mail addresses: huoxwu@xmu.edu.cn (H. Wu), dyyang@xmu.edu.cn (D. Yang), zjmath66@126.com (J. Zhang).

http://dx.doi.org/10.1016/j.jmaa.2016.05.044

$$\mathcal{O}(\mathcal{T}_*f)(x) := \Big(\sum_{i=1}^{\infty} \sup_{\epsilon_{i+1}\le t_{i+1}<t_i\le\epsilon_i} |T_{t_{i+1}}f(x) - T_{t_i}f(x)|^2\Big)^{1/2} \tag{1.2}$$

with $\{\epsilon_i\}$ being a fixed sequence decreasing to zero.

The L^p-boundedness of these operators were studied by Bourgain [8] for $p = 2$ and by Jones et al. [19] for $p \in [1,\infty)$ in the context of ergodic theory. Since then, in harmonic analysis, the study of boundedness of oscillation and variation operators associated with semigroups of operators and families of truncations of singular integrals have been paid more and more attention. In particular, Campbell et al. [10] first established the strong (p, p)-boundedness in the range $1 < p < \infty$ and the weak type $(1, 1)$-boundedness of the oscillation operator and the ρ-variation operator for the Hilbert transform. Subsequently, in [11], Campbell et al. further extended the results in [10] to the higher dimensional cases including Riesz transforms and general singular integrals with rough homogeneous kernels in $\mathbb{R}^d$. On the other hand, Jones and Reinhold [18] obtained the L^p-boundedness properties for $p \ge 1$ of the oscillation and variation operators associated with the symmetric diffusion semigroup (see Lemma 2.2). Crescimbeni et al. [14] subsequently established weighted variation inequalities of heat semigroup and Poisson semigroup associated to Laplacian and Hermite operator. For more results on oscillation and variation operators, we refer the readers to [7,9,15,20,22,27] and the references therein.

Let λ be a positive constant and $\triangle_\lambda$ be the Bessel operator which is defined by setting, for all suitable functions f on $\mathbb{R}_+ := (0,\infty)$,

$$\triangle_\lambda f(x) := -\frac{d^2}{dx^2}f(x) - \frac{2\lambda}{x}\frac{d}{dx}f(x).$$

An early work concerning the Bessel operator goes back to Muckenhoupt and Stein [23]. They developed a theory associated to $\triangle_\lambda$ which is parallel to the classical one associated to the Laplace operator $\triangle$. Since then, a lot of work concerning the Bessel operators was carried out; see, for example [1,2,4–6,16,21,25,26]. In particular, Betancor et al. in [3] established the characterizations of the atomic Hardy space $H^1((0,\infty), dm_\lambda)$ associated to $\triangle_\lambda$ in terms of the Riesz transform and the radial maximal function related to a class of functions including the Poisson semigroup $\{P_t^{[\lambda]}\}_{t>0}$ and the heat semigroup $\{W_t^{[\lambda]}\}_{t>0}$ as special cases, where $dm_\lambda(x) := x^{2\lambda}\,dx$ and dx is the Lebesgue measure.

The aim of this paper is to prove the $L^p(\mathbb{R}_+, dm_\lambda)$-boundedness and their endpoint estimates of the oscillation and variation operators for $\{P_t^{[\lambda]}\}_{t>0}$ and $\{W_t^{[\lambda]}\}_{t>0}$, respectively. To this end, we recall some necessary notation.

Let $P_*^{[\lambda]} := \{P_t^{[\lambda]}\}_{t>0}$ be a family of Poisson semigroup operators defined by

$$P_t^{[\lambda]}f(x) := e^{-t\sqrt{\triangle_\lambda}}f(x) = \int_0^\infty P_t^{[\lambda]}(x,y)f(y)\,y^{2\lambda}dy\ ,$$

where J_ν is the Bessel function of the first kind of order ν with $\nu \in (-1/2,\infty)$ and

$$\begin{aligned} P_t^{[\lambda]}(x,y) &= \int_0^\infty e^{-tz}(xz)^{-\lambda+1/2}J_{\lambda-1/2}(xz)(yz)^{-\lambda+1/2}J_{\lambda-1/2}(yz)\,dm_\lambda(z) \\ &= \frac{2\lambda t}{\pi}\int_0^\pi \frac{(\sin\theta)^{2\lambda-1}}{(x^2+y^2+t^2-2xy\cos\theta)^{\lambda+1}}\,d\theta,\ \ t,x,y\in(0,\infty); \end{aligned} \tag{1.3}$$

see [3].