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Optimal consumption-investment with critical wealth level

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ABSTRACT

We consider a consumption-investment problem in which the relevant economic conditions change, depending on whether the wealth exceeds a critical level or not. We propose a dynamic programming method to solve the problem by dividing the problem into subproblems split by wealth levels, and imposing a freeze condition at the boundaries. We then join the solutions of the subproblems so that the resulting value function is piecewise C^2. The methodology is illustrated through an application to a problem with nonnegative life insurance constraint.

1. Introduction

When the behavior of an agent changes abruptly – either voluntarily or involuntarily – we call it a phase transition. Transitions are often associated with some critical wealth levels. Bankruptcy is an obvious phase transition where consumption and investment activities come to a halt. Although zero wealth is a natural critical level, withdrawal from risky investments and implementation of austerity measures may occur when the wealth falls below a certain positive amount. In the other direction, when the wealth grows beyond a certain threshold, new investment opportunities may become available, and/or the investor may feel more comfortable with taking riskier opportunities. See [1,2,8].

We study optimal behavior of an agent when the relevant economic conditions change, depending on whether the current wealth exceeds a critical level or not. We allow the consumption preference to change as well. We first divide the problem into subproblems by wealth level in Section 2, and apply the standard dynamic programming approach in [3] to each subproblem. We find that when the wealth process is confined by a positive critical level, a non-monotonicity problem arises, and propose a modification to handle the problem.

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Section 3 deals with the main problem using the methodologies developed in Section 2. We conjecture a trial function by joining individual solutions, so that the resulting function is C^1 at the critical wealth level. We then verify that our trial function solves the Bellman equation and is indeed the value function.

We consider an application in Section 4. A simple life insurance model with nonnegative premium constraint, which is the first attempt to treat nonnegative life insurance within a consumption-investment framework. See [6] and [7].

2. Consumption-investment with freeze boundary

Using the mutual fund theorem (two fund separation theorem), we may assume there are two assets in the financial market: one is a risk-free asset, the other a risky asset (see [3]). We assume the risk free rate is a constant r and the risky asset's price process $(S_t)_{t\geq 0}$ follows a geometric Brownian motion, as in [3,4], and [5] etc.: $dS_t = S_t(\alpha\, dt + \sigma\, dw_t)$, where $(w_t)_{t\geq 0}$ a standard Brownian motion. Thus, the agent's wealth process $(x_t)_{t\geq 0}$ evolves according to $dx_t = (\mu\pi_t + rx_t - c_t)dt + \sigma\pi_t\, dw_t,\ t \geq 0$, where $\mu = \alpha - r$ is the excess rate of return of the risky asset, $c_t \geq 0$ and π_t the agent's consumption rate and investment in the risky asset, respectively, at time t. The agent chooses $c = (c_t)_{t\geq 0}$ and $\pi = (\pi_t)_{t\geq 0}$ dynamically to maximize his total utility from consumption in the presence of a critical wealth level $x_c \geq 0$, at which the agent is offered the option to cease all economic activities in exchange for the consolation value P in terms of remaining utility. Thus, the value function V is defined as

$$V(x_0) = V(x_0; x_c, P) = \sup_{(c_t,\pi_t)_{t\geq 0}} E\left[\int_0^\tau e^{-\beta t}U(c_t)\, dt + e^{-\beta\tau}P\right],$$

where $\beta > 0$ is the subjective discount rate, U the utility function from the instantaneous consumption which is strictly increasing and strictly concave, and τ the first time at which the agent wealth hits the critical level $x_c \geq 0$.

2.1. Downward freeze

Let the initial wealth be greater than the critical level, that is, $x_0 > x_c \geq 0$. The case where $x_c = 0$ is treated in [3] with the assumption that $U(0)/\beta \leq P < \lim_{c\to\infty} U(c)/\beta$. If the second inequality did not hold, then the value function would be identically equal to P but with no optimal policy, as explained in [3], since instantaneous bankruptcy is impossible. Therefore, we assume the second inequality holds too. Note that the lower bound P_* for the consolation value P in this case is the discounted utility for zero consumption stream: $P_* = U(0)/\beta = \int_0^\infty e^{-\beta t}U(0)dt$. If $P < U(0)/\beta$ in this case, the agent should behave as if P were $U(0)/\beta$ since this does not result in the wealth level touching $x_c = 0$, as explained in [3]. Some questions arise when $x_c > 0$: (a) What would the economically meaningful lower bound P_* be? (b) What would the policy of the agent be to avoid the freeze if $P < P_*$? Would it be feasible? (c) How does the optimal strategy change as the consolation value changes? Hereafter, we consider the case where $x_c > 0$ with the assumption that $P < \lim_{c\to\infty} U(c)/\beta$.

From the Bellman equation

$$\beta V(x) = \max_{c,\pi}\left[\mu\pi V'(x) + (rx - c)V'(x) + \frac{1}{2}\sigma^2\pi^2 V''(x) + U(c)\right], \tag{2.1}$$

we may derive a second order differential equation as in [3]

$$\gamma y^2 X''(y) + (\beta - r + 2\gamma)yX'(y) - rX(y) = -I(y) \tag{2.2}$$