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Nonlinear elliptic equations with a jumping reaction

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ABSTRACT

We study a nonlinear nonhomogeneous Dirichlet problem driven by the sum of a p-Laplacian and a Laplacian ($2 < p < +\infty$) and a jumping nonlinearity. Under very general conditions on the reaction and without using the Fučik spectrum, we show that the problem has at least three nontrivial solutions and we provide sign information for all of them. Our approach uses critical point theory, truncation and comparison techniques and Morse theory.

1. Introduction

Let $\Omega \in \mathbb{R}^N$ be a bounded domain with a C^2-boundary. In this paper, we study the following nonlinear nonhomogeneous Dirichlet problem:

$$\begin{cases} -\Delta_p u(z) - \Delta u(z) \;=\; f\big(z, u(z)\big) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \;\; 2 < p < +\infty. \end{cases} \tag{1.1}$$

Here $f(z,\zeta)$ is a measurable function which is C^1 in the ζ-variable and which exhibits asymmetric behaviour near $+\infty$ and $-\infty$. More precisely, we assume that the quotient $\frac{f(z,\zeta)}{|\zeta|^{p-2}\zeta}$ asymptotically at $+\infty$ stays below the principal eigenvalue $\widehat{\lambda}_1(p) > 0$ of $\big(-\Delta_p, W_0^{1,p}(\Omega)\big)$, while asymptotically at $-\infty$ stays above $\widehat{\lambda}_1(p)$. For both limits partial interaction with the eigenvalue is possible (nonuniform nonresonance). A reaction with this kind of behaviour is known in the literature as "jumping nonlinearity". In the past, problems with this kind of nonlinearities were studied in the context of p-Laplacian equations, using the so called "Fučik spectrum"

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(see Cuesta–de Figueiredo–Gossez [13], Motreanu–Tanaka [32], Tanaka [40]). There are two basic drawbacks in this approach. First, the use of the Fučik spectrum dictates that the limits $\lim\limits_{\zeta\to\pm\infty}\frac{f(z,\zeta)}{|\zeta|^{p-2}\zeta}$ exist. Second, the approach is constrained by our limited knowledge of the Fučik spectrum for the Dirichlet p-Laplacian. In fact, only the first nontrivial curve of the Fučik spectrum is known (see [13,40]). Our conditions on $f(z,\zeta)$ are more general and essentially our only asymptotic requirement at $\pm\infty$ is that as we move from $-\infty$ to $+\infty$, the quotient $\frac{f(z,\zeta)}{|\zeta|^{p-2}\zeta}$ crosses the principal eigenvalue $\widehat{\lambda}_1(p)$ ("jumping" or "crossing" nonlinearity). So, as we already said, at $+\infty$, the quotient $\frac{f(z,\zeta)}{|\zeta|^{p-2}\zeta}$ is below $\widehat{\lambda}_1(p)$, while at $-\infty$ is above $\widehat{\lambda}_1(p)$. Such an equation need not have negative solutions. In this paper, we show that problem (1.1) with such a "jumping nonlinearity" has at least three nontrivial solutions, one positive and two nodal (sign-changing). We should mention that for semilinear equations (i.e., $p=2$) with a reaction exhibiting a similar (symmetric) growth near $+\infty$ and near $-\infty$, first Hofer [24] obtained four nontrivial solutions, assuming that

$$f(z,\zeta)=f(\zeta)\in C^1(\mathbb{R}),\quad f'(0)\in\big(\widehat{\lambda}_m(2),\widehat{\lambda}_{m+1}(2)\big)\text{ for some } m\geqslant 2$$

(here $\big\{\widehat{\lambda}_k(2)\big\}_{k\geqslant 1}$ is the sequence of distinct eigenvalues of $\big(-\Delta,H^1_0(\Omega)\big)\big)$ and

$$\lim_{\zeta\to\pm\infty}\frac{f(\zeta)}{\zeta}\ <\ \widehat{\lambda}_1(2).$$

Two of these solutions have constant sign (one positive and the other negative). Later, it was recognized by Bartsch–Wang [4], that of the other two solutions, one is nodal (sign-changing). That in fact both are nodal, was proved by Dancer–Du [14] and Li–Wang [26]. Semilinear problems with asymmetric reaction were studied by Liu–Sun [29] and Motreanu–Motreanu–Papageorgiou [30], while p-Laplacian equations were studied by Motreanu–Motreanu–Papageorgiou [31] (Dirichlet problems) and Gasiński–Papageorgiou [19] (Neumann problems), with $f(z,\cdot)$ being $(p-1)$-superlinear near $+\infty$.

We mention that $(p,2)$-equations (i.e., equations in which the differential operator is the sum of a p-Laplacian and a Laplacian, $2<p<+\infty$) are important in quantum physics in the search for solitons, see Benci–D'Avenia–Fortunato–Pisani [5] and in plasma physics, see Cherfils–Il'yasov [9]. Recently such equations were studied by Aizicovici–Papageorgiou–Staicu [1], Cingolani–Degiovanni [10], Cingolani–Vannella [12], Gasiński–Papageorgiou [21,22], Mugnai–Papageorgiou [33], Papageorgiou–Rădulescu [35], Papageorgiou–Rădulescu–Repovš [36], Sun [38] and Sun–Zhang–Su [39]. In Aizicovici–Papageorgiou–Staicu [1] the authors produce nodal solutions for coercive problems. Cingolani–Degiovanni [10] and Cingolani–Vannella [12] consider nonresonant problems and prove only existence theorems. Gasiński–Papageorgiou [22] consider parametric equations and prove multiplicity theorems for large values of the parameter. Their problems are coercive. Gasiński–Papageorgiou [21] and Mugnai–Papageorgiou [33] study equations with a superlinear reaction term. Papageorgiou–Rădulescu [35] and Papageorgiou–Rădulescu–Repovš [36] deal with parametric equations near resonance. Finally Sun [38] and Sun–Zhang–Su [39] study coercive problems. Our problem here is noncoercive, asymmetric in the positive and negative directions and in the negative direction (that is at $-\infty$) it can be resonant with respect to any nonprincipal eigenvalue.

Our approach is variational based on the critical point theory, with parallel use of truncation and comparison techniques and Morse theory. In the next section, for the convenience of the reader, we recall the main mathematical tools which will be used in this work.

2. Mathematical background

In the analysis of problem (1.1), important is the Banach space

$$C^1_0(\overline{\Omega})\ =\ \big\{u\in C^1(\overline{\Omega}):\ u|_{\partial\Omega}=0\big\}.$$