Continued fraction inequalities related to $\left(1 + \frac{1}{x}\right)^x$

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A B S T R A C T

In this paper, we provide some new continued fraction inequalities related to $\left(1 + \frac{1}{x}\right)^x$.

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1. Introduction

Yang and Debnath [10] presented the following double inequality for every x in $0 < x \leq \frac{1}{5}$:

$$e - \frac{e}{2}x + \frac{11e}{24}x^2 - \frac{21e}{48}x^3 < (1+x)^{1/x} < e - \frac{e}{2}x + \frac{11e}{24}x^2. \tag{1.1}$$

Such inequalities were proven to be of great interest through the researchers, especially in the recent past, due to many practical problems where they can be applied. As example, we refer to inequality (1.1) which is the main tool for improving Carleman's inequality in [10].

Recently, Mortici and Yang [6] proposed an improvement of (1.1) and provided a simple, direct proof of (1.1) for every real number $x \in (0; 1]$.

$$a(x) < (1+x)^{1/x} < b(x), \tag{1.2}$$

where $a(x) = e - \frac{e}{2}x + \frac{11e}{24}x^2 - \frac{21e}{48}x^3 + \frac{2447e}{5760}x^4 - \frac{959e}{2304}x^5$ and $b(x) = a(x) + \frac{959e}{2304}x^5$.

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Motivated by the work of Yang and Debnath [10], Mortici and Yang [6], Hu and Mortici [5], in this paper we will continue our previous works, and apply the multiple-correction method to construct some new continued fraction inequalities related to $\left(1 + \frac{1}{x}\right)^x$, which have faster rate of convergence.

This paper is organized as follows. In Section 2, we explain how to find the continued fraction approximation for $\left(1 + \frac{1}{x}\right)^x$. In Section 3, we give the improved continued fraction approximation for $\left(1 + \frac{1}{x}\right)^x$.

Throughout the paper, the notation $\Psi(k; x)$ means a polynomial of degree k in x with all of its non-zero coefficients positive, which may be different at each occurrence.

2. Continued fraction approximation for $\left(1 + \frac{1}{x}\right)^x$

Theorem 1. *For every real number $x \geq 1$, we have*

$$\left(1 + \frac{1}{x}\right)^x < e \cdot a_i(x) = e \left(1 + \cfrac{s_1}{x + t_1 + \cfrac{s_2}{x+t_2+\cfrac{s_3}{x+t_3+\cfrac{s_4}{x+t_4+\cdots}}}}\right), i = 1, 2, 3, ..., \qquad (2.1)$$

where $s_1 = -\frac{1}{2}$, $t_1 = \frac{11}{12}$; $s_2 = -\frac{5}{144}$, $t_2 = \frac{34}{75}$; $s_3 = -\frac{481}{10000}$, $t_3 = \frac{357866}{757575}$; $s_4 = -\frac{792876605}{14692348944}$, $t_4 = \frac{2317657460602}{4805307952263}$; ....

Proof. Inequalities (2.1) are equivalent to $f_i < 0$ on $[1, +\infty)$, where

$$f_i(x) = x \ln\left(1 + \frac{1}{x}\right) - \ln a_i(x) - 1.$$

Based on our previous works [7,11–13], we will apply *multiple-correction method* to study the estimate for $\left(1 + \frac{1}{x}\right)^x$. Now, we prove the estimate for $\left(1 + \frac{1}{x}\right)^x$ by *multiple-correction method* [1–3] as follows:

(Step 1) The first-correction. Because $\left(x \ln\left(1 + \frac{1}{x}\right)\right)'' = -\frac{1}{x(1+x)^2}$, so we choose $a_1(x) = 1 + \frac{s_1}{x+t_1}$. Then letting the coefficient of x^4, x^3 of the molecule in the following fractions equal to zero, we have $s_1 = -\frac{1}{2}$, $t_1 = \frac{11}{12}$ and

$$f_1''(x) = \left(x \ln\left(1 + \frac{1}{x}\right) - \ln a_1(x) - 1\right)''$$

$$= \frac{P_1(x)}{x(1 + x)^2(5 + 12x)^2(11 + 12x)^2}.$$

As $P_1(x) = -3025 - 7296x - 4320x^2$ has all coefficients negative, it results that $f_1(x)$ is strictly concave on $[1, \infty)$ with $f_1(\infty) = 0$, we deduce that $f_1(x) < 0$ on $[1, \infty)$. As $f_1(1) = -1 + \ln\frac{23}{17} + \ln 2 = -0.00457195... < 0$.

(Step 2) The second-correction. We let $a_2(x) = 1 + \frac{s_1}{x+t_1+\frac{s_2}{x+t_2}}$. Then letting the coefficient of x^6, x^5 of the molecule in the following fractions equal to zero, we have $s_2 = -\frac{5}{144}$, $t_2 = \frac{34}{75}$ and

$$f_2''(x) = \left(x \ln\left(1 + \frac{1}{x}\right) - \ln a_2(x) - 1\right)''$$

$$= \frac{P_2(x)}{x(1 + x)^2(185 + 1044x + 1200x^2)^2(457 + 1644x + 1200x^2)^2}.$$

As $P_2(x) = -7147857025 - 52161296256x - 132444066720x^2 - 139358976000x^3 - 51948000000x^4$ has all coefficients negative, it results that $f_2(x)$ is strictly concave on $[1, \infty)$ with $f_2(\infty) = 0$, we deduce that $f_2(x) < 0$ on $[1, \infty)$. As $f_2(1) = -1 + \ln\frac{3301}{2429} + \ln 2 = -0.000107017... < 0$.