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On the asymptotic distribution of cranks and ranks of cubic partitions ☆

Byungchan Kim [a], Eunmi Kim [b,*], Hayan Nam [c]

[a] *School of Liberal Arts, Seoul National University of Science and Technology, 232 Gongneung-ro, Nowongu, Seoul 01811, Republic of Korea*
[b] *School of Mathematics, Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu, Seoul 02455, Republic of Korea*
[c] *Department of Mathematics, University of California, Irvine, CA 92697, United States*

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ABSTRACT

Cubic partitions are a special kind of bi-partitions whose name is inspired by a connection between a cubic continued fraction and an arithmetic property of this bi-partition. There are two partition statistics, namely rank and crank, for cubic partitions, which explain cubic partition congruences combinatorially. We obtain asymptotics for the number of cubic partitions of rank (resp. crank) m which reveal the distribution of the rank (resp. crank) among cubic partitions. As applications, we derive asymptotic inequalities between cubic partition rank and crank functions, and we remark upon the similarities and the differences between these and the corresponding inequalities for ordinary partitions.

1. Introduction

To find a combinatorial explanation for Ramanujan's striking three congruences of the partition function $p(n)$,

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11},$$

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* Corresponding author.
E-mail addresses: bkim4@seoultech.ac.kr (B. Kim), ekim@kias.re.kr (E. Kim), hayann@uci.edu (H. Nam).

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several partition statistics [3,11,13–15] have been introduced. Among these, the rank introduced by F.J. Dyson [11] and the crank defined by G.E. Andrews and F.G. Garvan [3] have been of greatest interest. Recent study of partition ranks and cranks has been focused on how they are distributed and what the differences between their distributions are (for the precise definitions of partition rank and crank, see Section 2). In this direction, Andrews, S.H. Chan, and the first author [2] proved that

$$\sum_{m\in\mathbb{Z}} m^k M(m,n) > \sum_{m\in\mathbb{Z}} m^k N(m,n) \tag{1.1}$$

for all positive integers k and n, where $M(m,n)$ (resp. $N(m,n)$) is the number of ordinary partitions of n with crank (resp. rank) m. As $\sum_{m\in\mathbb{Z}} M(m,n) = \sum_{m\in\mathbb{Z}} N(m,n) = p(n)$, this result implicitly says that it is likely that $M(m,n) > N(m,n)$ for relatively large m. On the other hand, K. Bringmann and J. Dousse [5] settled an old conjecture of Dyson; namely, under certain conditions on m,

$$M(m,n) = \frac{\pi}{4\sqrt{6n}} \operatorname{sech}^2\left(\frac{\pi m}{2\sqrt{6n}}\right) p(n) \left(1 + \mathcal{O}\left(\frac{m^{1/3}}{n^{1/4}}\right)\right).$$

More recently, Dousse and M. Mertens [10] showed that the same asymptotic formula holds for $N(m,n)$; i.e., the distribution for cranks and ranks are asymptotically similar when m is relatively smaller than n (more precisely, for $|m| \le \frac{\sqrt{n}\log n}{\pi\sqrt{6}}$ and actually this range covers almost all partitions [5, Corollary 1.3]).

In this paper, we are going to focus on the asymptotic distribution of two partition statistics, ranks and cranks for cubic partitions. Cubic partitions were introduced by H.-C. Chan in a series of papers [6–8]. H.-C. Chan showed that the cubic partition function $c(n)$ satisfies a Ramanujan-type congruence $c(3n+2) \equiv 0 \pmod 3$ [6] and further congruences modulo higher powers of 3 [7]. Analogous to Ramanujan's partition congruence modulo 5 derived from Rogers–Ramanujan continued fraction identities, a Ramanujan-type congruence for cubic partitions can be obtained from a cubic continued fraction identity, and this is the motivation for the name "cubic". The cubic partition function $c(n)$ is defined by

$$\sum_{n=0}^{\infty} c(n)q^n = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty}.$$

Here and in the rest of this paper, we will use the following standard q-series notations:

$$\begin{aligned}
(a;q)_0 &:= 1,\\
(a;q)_n &:= (1-a)(1-aq)\cdots(1-aq^{n-1}),\ n \ge 1,\\
(a;q)_\infty &:= \lim_{n\to\infty}(a;q)_n,\ |q|<1.
\end{aligned}$$

We can interpret $c(n)$ as the number of 2-color partitions of n with colors $r(ed)$ and $b(lue)$ subject to the restriction that the color b appears only in even parts. For example, there are four such partitions of 3:

$$3_r,\ \ 2_r + 1_r,\ \ 2_b + 1_r,\ \ 1_r + 1_r + 1_r.$$

Motivated by cubic partition congruences [6,7], the first author introduced a cubic partition crank which explains infinitely many congruences for powers of 3 explicitly [16] (see Section 2 for a definition). If we define $C(m,n)$ as the number of cubic partitions of n with crank m, then

$$\sum_{n=0}^{\infty}\sum_{m\in\mathbb{Z}} C(m,n)\zeta^m q^n = \frac{(q;q)_\infty (q^2;q^2)_\infty}{(\zeta q;q)_\infty (\zeta^{-1}q;q)_\infty (\zeta q^2;q^2)_\infty (\zeta^{-1}q^2;q^2)_\infty}. \tag{1.2}$$