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Extensions of Ramanujan's reciprocity theorem and the Andrews–Askey integral ☆

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A R T I C L E I N F O

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Dedicated to the memory of my parents

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A B S T R A C T

Ramanujan's reciprocity theorem may be considered as a three-variable extension of Jacobi's triple product identity. Using the method of q-partial differential equations, we extend Ramanujan's reciprocity theorem to a seven-variable reciprocity formula. Using the same method, the Andrews–Askey integral formula is extended to a q-integral formula which has seven parameters with base q.

1. Introduction and preliminaries

Throughout the paper we assume, unless otherwise stated, that $|q| < 1$ and use the standard product notation

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) \quad \text{and} \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

The q-binomial coefficients are the q-analogs of the binomial coefficients, which are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

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If n is an integer or ∞, the multiple q-shifted factorials are defined as

$$(a_1, a_2, ..., a_m; q)_n = (a_1; q)_n (a_2; q)_n \ldots (a_m; q)_n.$$

The Jacobi triple product identity is stated in the following proposition (see, for example [12, p. 1] and [14, p. 15]).

Proposition 1.1. *For $x \neq 0$, we have the triple product identity*

$$(q, x, q/x; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} x^n.$$

This identity is among the most important identities in mathematics, which has many interesting applications in number theory, combinatorics, analysis, algebra and mathematical physics. Some amazing extensions of this identity have been made by various authors. Ramanujan's ${}_1\psi_1$ summation formula and the Bailey ${}_6\psi_6$ summation formula both contain this identity as a special case, and may be considered as two important extensions of this identity, and these two extensions have wider applications than Jacobi's triple product identity.

Ramanujan's reciprocity theorem and the Andrews–Askey integral formula may also be regarded as two extensions of Jacobi's triple product identity. In this paper we will use the method of q-partial differential equations to extend Ramanujan's reciprocity theorem and Andrews–Askey integral formula to two more general q-formulae.

For simplicity, in this paper we use $\Delta(u, v)$ to denote the theta function

$$v(q, u/v, qv/u; q)_\infty. \tag{1.1}$$

As usual, the basic hypergeometric series or q-hypergeometric series ${}_r\phi_s$ is defined by

$${}_r\phi_s \left(\begin{matrix} a_1, ..., a_r \\ b_1, ..., b_s \end{matrix} ; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1, ..., a_r; q)_n}{(q, b_1, ..., b_s; q)_n} \left((-1)^n q^{n(n-1)/2} \right)^{1+s-r} z^n.$$

Now we introduce the definition of the Thomae–Jackson q-integral in q-calculus, which was introduced by Thomae [28] and Jackson [17].

Definition 1.2. Given a function $f(x)$, the Thomae–Jackson q-integral of $f(x)$ on $[a, b]$ is defined by

$$\int_a^b f(x) d_q x = (1-q) \sum_{n=0}^{\infty} [b f(bq^n) - a f(aq^n)] q^n.$$

If the function $f(x)$ is continuous on $[a, b]$, then, one can deduce that

$$\lim_{q \to 1} \int_a^b f(x) d_q x = \int_a^b f(x) dx.$$

In his lost notebook [26, p. 40], Ramanujan stated the following beautiful reciprocity theorem without proof. This formula may be considered as a three-variable extension of Jacobi's triple product identity. This result, now known as Ramanujan's reciprocity theorem, was first proved by Andrews in 1981 in his important paper [5]. For another proof, see Berndt et al. [10].