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The Schwarz lemma for functions with values in $C(V_{n,0})$ ☆

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A R T I C L E I N F O

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A B S T R A C T

In this paper, we first construct a type of Möbius transformations with Clifford coefficients and give some of its properties, for example: the mapping properties, preserving the symmetric points with respect to the sphere, the Jacobi determinant, the monogenic properties under these Möbius transformations. Then by using the integral representation formulas for harmonic functions with values in a Clifford algebra $C(V_{n,0})$ and the integral inequality estimation, we give the Schwarz lemma for harmonic functions with values in a Clifford algebra $C(V_{n,0})$, which is sharper than the known results. Combining the Schwarz lemma with the Möbius transformations, the Schwarz–Pick type lemmas for functions with values in $C(V_{n,0})$ are given. These results greatly improve the results in [21].

1. Introduction and preliminaries

Clifford analysis and quaternionic analysis were systematically studied in [4,9,11,13] etc. Developing the corresponding theories in Clifford analysis framework comparing with the theories in classical complex analysis is natural and interesting. The Schwarz lemmas play very important role in classical complex analysis. In [17], the Schwarz lemma in Clifford analysis was firstly considered, the result was built outside of the unit ball in $\mathcal{R}^{n+1}$. In [21], the Schwarz type lemmas inside the unit ball in $\mathcal{R}^{n+1}$ were built, the main tools are integral representations of harmonic functions and Möbius transformations with real coefficients in Clifford analysis. The Schwarz type lemmas in [17,21] are natural generalizations of the classical Schwarz lemma in complex analysis. Integral representation formulas and Möbius transformations were studied by many authors (see [1–3,5–8,10,12,14–22] etc.), they provide powerful tools for studying the Schwarz type lemmas in Clifford analysis setting. Since not all Clifford numbers are invertible, the general Möbius transformations with Clifford coefficients are not well-defined and more complicated. Moreover, the mapping properties of the general Möbius transformations with Clifford coefficients are not well established. In [21], the Schwarz–Pick type lemmas were built based on the Möbius transformations with real coefficients, it is worthwhile to

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make the inequality estimation sharper and improve it by using the more general Möbius transformations with Clifford coefficients. With this motivation, in this paper, by constructing the corresponding type of Möbius transformations with no longer real coefficients but Clifford coefficients, we give some properties of these Möbius transformations. Combining the integral representation formulas, the integral inequality estimation with the Möbius transformations, the Schwarz–Pick type lemmas for functions with values in a Clifford algebra $C(V_{n,0})$ are presented.

Let $V_{n,0}$ be an n-dimensional ($n \geq 1$) real linear space with basis $\{e_1, e_2, \cdots, e_n\}$, $C(V_{n,0})$ be the 2^n-dimensional real linear space with basis

$$\{e_A, A = \{h_1, \cdots, h_r\} \in \mathcal{P}N, 1 \leq h_1 < \cdots < h_r \leq n\},$$

where N stands for the set $\{1, \cdots, n\}$ and $\mathcal{P}N$ denotes the family of all order-preserving subsets of N in the above way. We denote $e_\emptyset$ as e_0 and e_A as $e_{h_1\cdots h_r}$ for $A = \{h_1, \cdots, h_r\} \in \mathcal{P}N$. The product on $C(V_{n,0})$ is defined by

$$\begin{cases} e_A e_B = (-1)^{\#(A\cap B)}(-1)^{P(A,B)} e_{A\triangle B}, & \text{if } A, B \in \mathcal{P}N, \\ \lambda\mu = \sum\limits_{A\in\mathcal{P}N} \sum\limits_{B\in\mathcal{P}N} \lambda_A \mu_B e_A e_B, & \text{if } \lambda = \sum\limits_{A\in\mathcal{P}N} \lambda_A e_A,\ \mu = \sum\limits_{B\in\mathcal{P}N} \mu_B e_B, \end{cases} \tag{1.1}$$

where $\#(A)$ is the cardinal number of the set A, the number $P(A, B) = \sum\limits_{j\in B} P(A, j)$, $P(A, j) = \#\{i, i \in A, i > j\}$, the symmetric difference set $A\triangle B$ is also order-preserving in the above way, and $\lambda_A \in \mathcal{R}$ is the coefficient of the e_A-component of the Clifford number λ. We also denote λ_0 as $\mathrm{Re}(\lambda)$. Thus $C(V_{n,0})$ is called the Clifford algebra over $V_{n,0}$.

An involution is defined by

$$\begin{cases} \overline{e_A} = (-1)^{\sigma(A)} e_A, & \text{if } A \in \mathcal{P}N, \\ \overline{\lambda} = \sum\limits_{A\in\mathcal{P}N} \lambda_A \overline{e_A}, & \text{if } \lambda = \sum\limits_{A\in\mathcal{P}N} \lambda_A e_A, \end{cases} \tag{1.2}$$

where $\sigma(A) = \#(A)(\#(A) + 1)/2$. The operator D and $\overline{D}$ are written as

$$D = \sum_{k=0}^{n} e_k \frac{\partial}{\partial x_k} : C^{(r)}(\Omega, C(V_{n,0})) \to C^{(r-1)}(\Omega, C(V_{n,0})),$$

and

$$\overline{D} = \sum_{k=0}^{n} \overline{e_k} \frac{\partial}{\partial x_k} : C^{(r)}(\Omega, C(V_{n,0})) \to C^{(r-1)}(\Omega, C(V_{n,0})).$$

2. Some lemmas and Möbius transformation

The real linear space with basis $\{e_0, e_1, \cdots, e_n\}$ is a subspace of $C(V_{n,0})$, which is denoted by Cl_{n+1}. $Cl_{n+1} = \{\mathbf{x} = x_0 + x_1 e_1 + \cdots + x_n e_n : x_0, x_1, \cdots, x_n \in \mathcal{R}\}$, Cl_{n+1} is identical with the Euclidean space $\mathcal{R}^{n+1}$. Denote $\partial B(\mathbf{x}, r) = \{\mathbf{y} \in Cl_{n+1} \| \mathbf{y} - \mathbf{x}| = r\}$, then $\partial B(0, 1)$ is the unit sphere in hyper-complex space Cl_{n+1}. Suppose Ω and Ω^* are open, non-empty subsets of Cl_{n+1}, $\Omega \subset \Omega^*$. Let f be a function with value in $C(V_{n,0})$ defined in Ω^*.

Lemma 2.1. *(See [21].) Suppose $f \in C^2(\Omega^*, C(V_{n,0}))$, $\triangle[f] = 0$ in Ω^*. Then for $\forall \mathbf{x} \in B(\mathbf{a}, R) \subset \Omega^*$,*