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Sequence-singular operators

Gleb Sirotkin *, Ben Wallis

Department of Mathematical Sciences, Northern Illinois University, DeKalb, IL 60115, United States

A R T I C L E I N F O

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A B S T R A C T

In this paper we study ℓ_p-related collections of operators introduced by Beanland and Freeman in [6], on the subject of forming operator ideals. We show that these collections are not always closed under addition, and hence do not form operator ideals. Nevertheless, they allow us to construct an uncountable chain of closed ideals in each of the operator algebras $\mathcal{L}(\ell_1 \oplus \ell_q)$, $1 < q < \infty$, and $\mathcal{L}(\ell_1 \oplus c_0)$. This finishes answering a longstanding question of Pietsch.

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1. Introduction

For many years, researchers have been interested in discovering whether or not, given a particular Banach space X, the operator algebra $\mathcal{L}(X)$ admits infinitely many closed ideals. In the case of many classical Banach spaces, this has long been decided. For instance, in 1960 it was shown that $\mathcal{L}(\ell_p)$, $1 \leq p < \infty$, and $\mathcal{L}(c_0)$ admit exactly three closed ideals [7]. This also took care of the case $\mathcal{L}(L_2)$, since $L_2 \cong \ell_2$. By 1978 it was discovered that $\mathcal{L}(L_p)$ admits infinitely many closed ideals for $p \in (1, 2) \cup (2, \infty)$ [11, Theorem 5.3.9], which in 2014 was improved to show continuum many (follows from [14]). Also in 1978 it was shown that $\mathcal{L}(C[0, 1])$ admits uncountably many closed ideals [11, Theorem 5.3.11]. Whether $\mathcal{L}(L_1)$ and $\mathcal{L}(L_\infty) \cong \mathcal{L}(\ell_\infty)$ admit infinitely many closed ideals remains a significant open question.

Besides these classical cases, the closed ideal structures of $\mathcal{L}(\ell_p \oplus \ell_q)$, $1 \leq p < q < \infty$, have generated a great deal of interest. Although Pietsch asked as early as 1978 whether these operator algebras admit infinitely many closed ideals [11, Problem 5.33], the question remained entirely open for over 36 years. Indeed, not until 2014 was it finally shown that $\mathcal{L}(\ell_p \oplus \ell_q)$ admits continuum many closed ideals whenever $1 < p < q < \infty$ [14]. Then, in 2015 it was shown that this result extends to $\mathcal{L}(\ell_p \oplus c_0)$ and $\mathcal{L}(\ell_1 \oplus \ell_q)$ in the special cases $1 < p < 2 < q < \infty$ [16, Theorem 1.1]. As the main result of this paper, we close Pietsch's question by proving the following.

* Corresponding author.
E-mail addresses: gsirotkin@niu.edu (G. Sirotkin), bwallis@niu.edu (B. Wallis).

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1.1 Theorem. *Algebras $\mathcal{L}(\ell_1 \oplus \ell_q)$, $1 < q \leq \infty$, and $\mathcal{L}(\ell_1 \oplus c_0)$ each admit an uncountable chain of closed ideals.*

Unfortunately, the cases $\mathcal{L}(\ell_p \oplus c_0)$ fail to dualize, and remain open for $2 \leq p < \infty$.

To prove Theorem 1.1 we will use recently introduced by Beanland and Freeman [6] classes $\mathcal{WS}_{e,\xi}$. Fix a seminormalized basis $e = (e_n)$ for a Banach space E. We say that an operator $T \in \mathcal{L}(X, Y)$, X and Y Banach spaces, is **(e_n)-singular** if for every normalized basic sequence (x_n) in X, the image sequence (Tx_n) fails to dominate (e_n). We denote by $\mathcal{WS}_{e,\omega_1}(X, Y)$ the class of all (e_n)-singular operators in $\mathcal{L}(X, Y)$. In [6, Proposition 2.8] the following interesting results were proved about class $\mathcal{WS}_{e,\omega_1}$ for certain nice choices of e.

- If $e = (e_n)$ denotes the canonical basis for c_0 then $\mathcal{WS}_{e,\omega_1} = \mathcal{K}$, the compact operators.
- If $e = (e_n)$ denotes the summing basis for c_0 then $\mathcal{WS}_{e,\omega_1} = \mathcal{W}$, the weakly compact operators.
- If $e = (e_n)$ denotes the canonical basis for ℓ_1 then $\mathcal{WS}_{e,\omega_1} = \mathcal{R}$, the Rosenthal operators.

(Recall that an operator $T \in \mathcal{L}(X, Y)$ is **Rosenthal** if for every bounded sequence (x_n) in X, (Tx_n) admits a weak Cauchy subsequence.) Each of these classes is a norm-closed operator ideal, and so it is natural to conjecture that the class $\mathcal{WS}_{e,\omega_1}$ could also form an operator ideal for other nice choices of $e = (e_n)$. In particular, we might expect $\mathcal{WS}_{e,\omega_1}$ to be an operator ideal whenever $e = (e_n)$ is the canonical basis of ℓ_p, $1 < p < \infty$.

As our first result, we show in Section 2 that the above conjecture is false, as for any $1 < p < \infty$ we can choose spaces X and Y such that $\mathcal{WS}_{e,\omega_1}(X, Y)$ fails to be closed under addition when $e = (e_n)$ is the canonical basis for ℓ_p, $1 < p < \infty$.

Despite this, the ideas originating from [3] allow us to use $\mathcal{WS}_{e,\xi}$ to show in Section 3 that $\mathcal{L}(\ell_1 \oplus \ell_q)$, $1 < q < \infty$, and $\mathcal{L}(\ell_1 \oplus c_0)$ each admit an uncountable chain of closed ideals. This is especially significant since it represents the last ingredient needed to answer a longstanding open question of Pietsch [11, Problem 5.33].

For the most part, all definitions and notation are standard, as are found, for instance, in [2]. However, we will restate some of the most important ones here. Let $\mathcal{J}$ be a subclass of the class $\mathcal{L}$ of all continuous linear operators between Banach spaces, and if X and Y are Banach spaces then we write $\mathcal{J}(X, Y) = \mathcal{L}(X, Y) \cap \mathcal{J}$, a component. We say that $\mathcal{J}$ has the **ideal property** whenever $BTA \in \mathcal{J}(W, Z)$ for all $A \in \mathcal{L}(W, X)$, $B \in \mathcal{L}(Y, Z)$, and $T \in \mathcal{J}(X, Y)$, and all Banach spaces W, X, Y, and Z. If in addition every component $\mathcal{J}(X, Y)$ is a linear subspace of $\mathcal{L}(X, Y)$ containing all the finite-rank operators therein, then $\mathcal{J}$ is an **operator ideal**. We say that $\mathcal{J}$ is norm-closed (closed under addition) whenever all its components $\mathcal{J}(X, Y)$ are norm-closed (closed under addition) in $\mathcal{L}(X, Y)$. Let us also borrow a piece of terminology from [13]: If X and Y are Banach spaces, then a linear subspace $\mathcal{J}$ of $\mathcal{L}(X, Y)$ is called a **subideal** if whenever $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$, and $T \in \mathcal{J}$, we have $BTA \in \mathcal{J}$. A subideal of $\mathcal{L}(X)$ is called, simply, an **ideal**. (For operator algebras, this coincides with the notion of an *ideal* in the algebraic sense.)

If M is an infinite subset of $\mathbb{N}$, then denote by $[M]$ the family of all infinite subsets of M, and denote by $[M]^{<\omega}$ the family of all finite subsets of M. For $n \in \mathbb{N}$ let $[M]^{\leq n} = \{A \in [M]^{<\omega} : \#A \leq n\}$, i.e. the family of all subsets of M of size $\leq n$. If $\mathcal{F}$ is a subset of $[\mathbb{N}]^{<\omega}$ and $M = (m_i) \in [\mathbb{N}]$, then we define

$$\mathcal{F}(M) = \{(m_i)_{i \in E} : E \in \mathcal{F}\}.$$

If $\mathcal{F}$ and $\mathcal{G}$ are both subsets of $[\mathbb{N}]^{<\omega}$ then we define

$$\mathcal{F}[\mathcal{G}] = \left\{ \bigcup_{i=1}^{n} E_i : E_1 < \cdots < E_n, E_i \in \mathcal{G} \forall i, (\min E_i)_{i=1}^{n} \in \mathcal{F} \right\}.$$

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