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Locally recurrent functions, density topologies and algebraic genericity

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ABSTRACT

We study the algebraic and topological genericity of certain subsets of locally recurrent functions, obtaining (among other results) algebrability and spaceability within these classes.

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1. Introduction

This paper contributes to the ongoing search for linear structures of mathematical objects enjoying certain special or *unexpected* properties. This search began at the beginning of this century and, since then, many authors have become interested in this direction of research. Many different fields in Mathematics were influenced by this, from Linear Chaos to Real and Complex Analysis, passing through Set Theory and Linear and Multilinear Algebra, or even Operator Theory, Topology, Measure Theory, Abstract Algebra and Probability Theory. We refer the interested reader to the recent survey paper [10] and to the monograph [1] for a full detailed study of this modern area of research.

Let us now present some, by now, well known definitions on algebraic genericity (see, e.g. [1,9,10,23,37]).

Definition 1.1. Let κ be a cardinal number.

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i) Let L be a vector space and a set $A \subset L$. We say that A is *κ-lineable* if $A \cup \{0\}$ contains a κ-dimensional vector space.
ii) Let L be a Banach space and a set $A \subset L$. We say that A is *κ-spaceable* if $A \cup \{0\}$ contains a κ-dimensional closed vector space. (We say that A is *spaceable* if A is κ-spaceable for some infinite κ.)
iii) Let L be a linear commutative algebra and a set $A \subset L$. We say that A is *κ-algebrable* if $A \cup \{0\}$ contains a κ-generated algebra B (i.e. there exists a minimal system of generators of B with cardinality κ).
iv) Let L be a linear commutative algebra and a set $A \subset L$. We say that A is *κ-strongly algebrable* if $A \cup \{0\}$ contains a κ-generated algebra B that is isomorphic with a free algebra (i.e. if $X = \{x_\alpha, a < \kappa\}$ denotes the set of generators of B then the set $\widetilde{X}$ of all elements of the form $x_{\alpha_1}^{k_1} x_{\alpha_2}^{k_2} \cdots x_{\alpha_n}^{k_n}$ is linearly independent and all linear combinations of the elements from $\widetilde{X}$ are in $A \cup \{0\}$).

This article is arranged in two main sections. Section 2 focuses on continuous locally recurrent functions and its algebraic and topological genericity within subsets of continuous functions, obtaining spaceability and strong algebrability within this class of functions. Section 3 deals with lineability, algebrability and spaceability results within classes of locally recurrent functions that are continuous with respect to certain density topologies.

2. Strong algebrability and spaceability of continuous locally recurrent functions

The following definition was first introduced in [12] (see also [8]).

Definition 2.1. Let $I \subset \mathbb{R}$ be a non-trivial closed interval and let $x \in I$. A function $f : I \to \mathbb{R}$ is said to be *right (left) recurrent* at x if, given any $\varepsilon > 0$, there exists $y \in I$ such that $0 < y - x < \varepsilon$ $(0 < x - y < \varepsilon)$ and $f(y) = f(x)$.

The function f is called *locally recurrent* on I, if it is (left or right) recurrent at each $x \in I$. We will denote the set of all such functions by $\mathsf{LR}(I)$.

The property of being locally recurrent is linked to *everywhere surjectivity*:

Definition 2.2 *([2])*. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *everywhere surjective* ($f \in \mathsf{ES}(\mathbb{R})$ from now on) if, given any interval $I \subset \mathbb{R}$, $f(I) = \mathbb{R}$.

It is clear that any everywhere surjective function is locally recurrent (on any non-trivial subinterval of $\mathbb{R}$). Furthermore, the class of everywhere surjective functions enjoys the following interesting properties: Any $f \in \mathbb{R}^{\mathbb{R}}$ can be expressed as both, a sum of two everywhere surjective functions and a limit of a sequence of everywhere surjective functions (see [25,38]). Furthermore, in [2] the authors show that the class $\mathsf{ES}(\mathbb{R})$ (and therefore the class $\mathsf{LR}(\mathbb{R})$) is $2^{\mathfrak{c}}$-lineable, which is the best possible result in terms of dimension (see [1,10,14,26] for more lineability-related results on everywhere surjective functions).

In the rest of this section we shall focus on a more concrete subset of LR functions. Namely, we study the set of non-constant, continuous, locally recurrent functions with derivative zero almost everywhere. The first construction of such a function appeared in [34]. While the original construction relies heavily on the properties of decimal expansion, we present another construction that only requires the use of elementary properties of real functions and, in our opinion, allows for easier verification of local recurrence. This construction first appeared in [30, Theorem 23.16, p. 179]. Since the original source is not readily accessible, we present it here in detail.

Theorem 2.3 *([30], Theorem 23.16, p. 179). There exists a function $f : [0, 1] \to \mathbb{R}$ such that f is non-constant, continuous, locally recurrent and $f'(x) = 0$ for almost every $x \in [0, 1]$.*

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