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# Principal solutions at infinity for time scale symplectic systems without controllability condition

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ABSTRACT

In this paper we introduce a new concept of a principal solution at infinity for nonoscillatory symplectic dynamic systems on time scales. The main ingredient is that we avoid the controllability (or normality) condition, which is traditionally assumed in this theory in the current literature. We show that the principal solutions at infinity can be classified according to the eventual rank of their first component and that the principal solutions exist for all values of the rank between explicitly given minimal and maximal values. The minimal value of the rank is connected with the eventual order of abnormality of the system and it gives rise to the so-called minimal principal solution at infinity. We show that the uniqueness property of the principal solutions at infinity is satisfied only by the minimal principal solution. In this study we unify and extend to arbitrary time scales the recently introduced theory of principal and recessive solutions at infinity for possibly abnormal (continuous time) linear Hamiltonian differential systems and (discrete time) symplectic systems. Moreover, the new theory on time scales also shows that in some results from the continuous time theory the needed assumptions can be simplified.



## 1. Introduction

In this paper we introduce a new theory of principal solutions at infinity (sometimes also called recessive solutions or minimal solutions or distinguished solutions at infinity) for symplectic dynamic systems on time scales

$$x^{\Delta} = \mathcal{A}(t)\,x + \mathcal{B}(t)\,u, \quad u^{\Delta} = \mathcal{C}(t)\,x + \mathcal{D}(t)\,u, \quad t \in [a,\infty)_{\mathbb{T}}. \tag{S}$$

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Here $\mathbb{T}$ is a time scale, i.e., a nonempty closed subset of $\mathbb{R}$, which is unbounded from above and bounded from below with $a := \min \mathbb{T}$, and $[a,\infty)_{\mathbb{T}} := [a,\infty) \cap \mathbb{T}$. The coefficients $\mathcal{A}(t)$, $\mathcal{B}(t)$, $\mathcal{C}(t)$, $\mathcal{D}(t)$ are real piecewise rd-continuous $n \times n$ matrices on $[a,\infty)_{\mathbb{T}}$ such that

$$\mathcal{S}^T(t)\,\mathcal{J} + \mathcal{J}\mathcal{S}(t) + \mu(t)\,\mathcal{S}^T(t)\,\mathcal{J}\mathcal{S}(t) = 0, \quad t \in [a,\infty)_{\mathbb{T}}, \tag{1.1}$$

where $\mu(t)$ is the graininess of $\mathbb{T}$ and

$$\mathcal{S}(t) := \begin{pmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ \mathcal{C}(t) & \mathcal{D}(t) \end{pmatrix}, \quad \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

For the basic theory of dynamic equations on time scales and symplectic systems on time scales we refer to [6,7,14] and [1,11,12,15–17].

According to [10, Section 3], a conjoined basis $(\hat{X},\hat{U})$ of (S) is said to be a principal solution at infinity if $\hat{X}(t)$ is invertible for large $t$, the matrix $[\hat{X}^\sigma(t)]^{-1}\mathcal{B}(t)\,[\hat{X}^T(t)]^{-1} \geq 0$ (i.e., it is positive semidefinite) for large $t$, and

$$\lim_{t\to\infty} X^{-1}(t)\,\hat{X}(t) = 0 \tag{1.2}$$

for any conjoined basis $(X,U)$ of (S) for which the (constant) Wronskian matrix $N := X^T\hat{U} - U^T\hat{X}$ is nonsingular. Here $\sigma(t)$ is the forward jump operator at $t$ and $f^\sigma(t) := f(\sigma(t))$. The main results in [10, Theorems 3.1 and 3.3] then state that if the system (S) is nonoscillatory and satisfies a certain normality (or controllability) condition, then (S) has a principal solution $(\hat{X},\hat{U})$ at infinity, this principal solution is unique up to a constant invertible multiple, and it is characterized by the integral criterion

$$\lim_{t\to\infty} \Big( \int^t [\hat{X}^\sigma(s)]^{-1}\mathcal{B}(s)\,[\hat{X}^T(s)]^{-1}\Delta s \Big)^{-1} = 0. \tag{1.3}$$

The normality condition means that there exists $T \in [a,\infty)_{\mathbb{T}}$ such that if a solution $(x,u)$ of (S) has $x(t) = 0$ on a nondegenerate subinterval of $[T,\infty)_{\mathbb{T}}$, then also $u(t) = 0$ on $[T,\infty)_{\mathbb{T}}$. Oscillation properties of the principal solution of (S) were investigated e.g. in [4,5,10].

The above definition of the principal solution at infinity covers the corresponding notions for the classical continuous time linear Hamiltonian systems

$$z' = \mathcal{J}\,\mathcal{H}(t)\,z, \quad \mathcal{H}^T(t) = \mathcal{H}(t), \quad t \in [a,\infty), \tag{1.4}$$

defined by Reid, Hartman, or Coppel in [9,13,20], and for discrete symplectic systems

$$z_{k+1} = \mathcal{S}_k z_k, \quad \mathcal{S}_k^T \mathcal{J}\mathcal{S}_k = \mathcal{J}, \quad k \in [0,\infty)_{\mathbb{Z}}, \tag{1.5}$$

where $[0,\infty)_{\mathbb{Z}} := [0,\infty) \cap \mathbb{Z}$, defined by Došlý or Ahlbrandt and Peterson in [2,10]. It is known in the above mentioned references that the principal solution at infinity of systems (S), resp. (1.4) or (1.5), possesses certain extremal properties, especially in the relation with the associated Riccati matrix equation, and that it has many applications in the oscillation and spectral theory.

The aim of this paper is to develop the theory of principal solutions at infinity for system (S) when no normality condition is assumed. Our recent studies in the continuous time systems [21,22] and discrete time systems [23] indicate, how to avoid the inverses in (1.3) by employing the Moore–Penrose pseudoinverses. Therefore, we extend and unify the principal solutions at infinity for systems (1.4) and (1.5) from [21,22] and [23] into a single notion on arbitrary time scales (Definition 6.1). This allows to explain the differences in the continuous and discrete time definitions, as well as in the main results of [21,22] and [23].