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On Hadamard well-posedness of families of Pareto optimization problems

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ABSTRACT

This paper deals with the well-posedness of families of finite dimensional vector optimization problems ordered by components (Pareto problem). For this problem, two Hadamard well-posedness concepts are introduced. These concepts involve the existence and uniqueness of efficient/weak efficient solutions, and also the continuous behavior of these solutions with respect to perturbations of the data. The perturbations in the last property are formulated through a variational convergence notion of the objective functions and by considering approximate solutions of the perturbed problems. Necessary and sufficient conditions for the well-posedness of Pareto optimization problems are obtained in general, and also under convexity and quasiconvexity assumptions. To do this, asymptotic mathematical tools are employed. Finally, it is proved that the convex Pareto optimization problems are "essentially" well-posed in the sense of category theory.

1. Introduction

The notion of well-posedness of a given problem is crucial in the theory of variational problems, since it plays an important role in sensitivity and convergence analysis of numerical methods. Hadamard [15] first introduced a notion of well-posedness for studying mathematical models of physical phenomena. This notion requires existence and uniqueness of solutions of the models, together with continuous dependence on the data. At present this notion is termed Hadamard well-posedness (for short, H.w.p.). It is closely related to the stability of the model.

In the sixties, Tykhonov [27] introduced a notion of well-posedness for scalar optimization problems. A scalar minimization problem is said to be Tykhonov well-posed if it has a unique solution toward which every minimizing sequence of the problem converges, and it is said to be Tykhonov well-posed in the generalized sense if it has solutions and every minimizing sequence of the problem has a subsequence

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converging toward a minimizer. Tykhonov well-posedness can be used for studying these problems only in an abstract way, since it does not take into account that several algorithms for constrained optimization problems provide approximate solutions which do not lie in the feasible set, but get closer to it. Thus, it makes sense to consider also minimizing sequences that are close to the feasible set. This observation gives rise to a notion of well-posedness that was defined by Levitin–Polyak [16] in the sixties.

A more recent well-posedness concept is the so-called well-posedness by perturbations or extended well-posedness. This notion was introduced by Zolezzi [29] and it is important since it unifies Tykhonov well-posedness and H.w.p., allowing perturbations of the objective function. The literature on well-posedness for scalar optimization problems is rich, for instance, one has the books of Dontchev–Zolezzi [7] and Lucchetti [22].

The study of well-posedness for vector optimization problems is not so advanced as for the scalar case, and several definitions have been proposed. In [21] one can find a review of this topic concerning the eighties and nineties. At present, there is not a commonly accepted definition of well-posedness for these problems, since there are plenty of notions of approximate solution and, thus, it is not easy to choose a convenient concept for minimizing sequences.

In this paper, we employ our recent notion of approximate solution from [13] to introduce and to study notions of H.w.p. for families of Pareto optimization problems. We do this, since it includes as particular cases the most important concepts of approximate solution for Pareto optimization problems of the literature. Moreover, we employ the variational convergence notion from [19,20] to perturb Pareto optimization problems. As a consequence, our notions of H.w.p. differ from those in the literature (see [3,5,28,17]) in two aspects. First, by H.w.p. they mean only the continuous dependence from the data; i.e., the requirement of the existence of solutions is ignored. Second, they define H.w.p. of a given problem whereas we define it for a family of problems. To deal with the existence of solutions and the continuous dependence from the data we employ an asymptotic method that allows us to study them simultaneously.

This work is structured as follows. In Section 2, the framework of the paper and some mathematical tools and notations needed along it are introduced. In Section 3, the stability of the sets of solutions involved in this paper is studied. In particular, some conditions are stated for guaranteeing the outer semicontinuity of the solution-set-mapping of approximate solutions with respect to the efficient and weak efficient sets when the approximate error tends to zero and the perturbed problems tend to the nominal problem. In Section 4, two H.w.p. concepts for families of Pareto optimization problems are defined and characterized through the limit behavior of the approximate solutions of Pareto problems near to the nominal problem, when both the precision error of the approximate solutions and the distance from the Pareto problems to the nominal problem tend to zero. We underline that both concepts can be applied to study the H.w.p. not only for the weak efficient solutions, which is usual for this kind of results in the literature, but also for the efficient solutions, which is addressed here for the first time in the literature to the best of our knowledge. In Section 5, some asymptotic conditions for studying the H.w.p. of convex and quasiconvex Pareto optimization problems are derived. As a consequence, it is proved that convex Pareto optimization problems are essentially H.w.p. in the sense of category theory.

2. Notation and preliminaries

In this work, $\mathbb{R}^m_+$ denotes the nonnegative orthant of $\mathbb{R}^m$. We denote $\mathbb{R}_+ := \mathbb{R}^1_+$, $\mathbb{R}_- := -\mathbb{R}_+$, $\mathbb{R}^m_{+0} := \mathbb{R}^m_+\backslash\{0\}$ and $\mathbb{R}^m_{++} := \operatorname{int}\mathbb{R}^m_+ \cup \{0\}$. For a set $M \subset \mathbb{R}^m$, we denote by $\operatorname{cl} M$ its closure, by $\operatorname{int} M$ its topological interior, by $\operatorname{cone} M$ the cone that it generates and by $\operatorname{cone}_+ M := \{\alpha y : \alpha > 0, y \in M\}$ the positive cone that it generates. We say that M is coradiant if $\alpha d \in M$ for all $d \in M$ and $\alpha > 1$, M is free-disposal with respect to $\mathbb{R}^m_{+0}$ if $M + \mathbb{R}^m_{+0} = M$ and M is pointed if $M \cap (-M) \subset \{0\}$.

We assume that $\mathbb{R}^m$ is partially ordered via the relation $\leq_{\mathbb{R}^m_+}$ defined for $y, z \in \mathbb{R}^m$ by $y \leq_{\mathbb{R}^m_+} z$ iff $z - y \in \mathbb{R}^m_+$. Analogously, we will also consider the relation $<_{\mathbb{R}^m_{++}}$ defined for $y, z \in \mathbb{R}^m$ by $y <_{\mathbb{R}^m_{++}} z$ iff