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# Local phase synchronization and clustering for the delayed phase-coupled oscillators with plastic coupling

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ABSTRACT

In this paper, we investigate the local synchronization problem for delayed phase-coupled oscillators with plastic coupling (also referred as a generalized Kuramoto model). By linearizing the phase-coupled oscillators system at a special solution, an approximation phase-coupled system with delay effects is deduced. Moreover, we find the multiplicity of semisimple zero eigenvalue for coupling strength coefficients matrix and the rank of Vandermonde matrix associated initial values are two sensitive factors. The former is closely related to the maximum number of clusters phase of linear coupled system, and the latter determines clusters phase synchronization. As results, both frequency synchronization criteria and phase synchronization criteria are established in this literature. Also, the final synchronization frequency and phase are formulated in terms of coupling strength, initial values and time delay.



## 1. Introduction

The synchronization problem of phase-coupled oscillators has been widely studied in different disciplines ranging from physics [5,20], networks [4,10,19] and engineering [7,16]. Synchronization of Kuramoto oscillators (a classical phase-coupled system), for example, has been extensively analyzed under various assumptions on homogeneous oscillators and network topology. In particularly, Luca [18] conducted an interesting study on synchronization and stability for systems of homogeneous oscillators with plastic coupling, but the study focused on complete graph topology. Recently, Gushchin–Mallada–Tang [8] discussed a system of phase-coupled oscillators with plastic coupling governed by the following two classes of equations:

$$\dot{\theta}_i(t) = \omega_i + \sum_{j\in N_i} k_{ij} \cdot f_{ij}(\theta_j(t) - \theta_i(t)),\ 1 \le i \le n, \tag{1}$$

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$$\dot{k}_{ij}(t) = s_{ij}(\alpha_{ij}F_{ij}(\theta_j(t) - \theta_i(t)) - k_{ij}),\ (i,j) \in E, \tag{2}$$

where $\theta_i$ is the phase and $\omega_i$ the intrinsic frequency of oscillator $i$, $E$ is the set of edges, $N_i$ is a set of oscillators connected to oscillator $i$ (i.e. the set of its neighbors), $k_{ij}$ and $f_{ij}$ which are $2\pi$-periodic continuously differentiable functions. $F_{ij}(x) = -\int_0^x f_{ij}(t)dt + C$ with the integration constant $C$ chosen so that $\int_0^\pi F_{ij}(t)dt = 0$. Positive constants $\alpha_{ij}$ determine minimum and maximum values of the coupling strengths, and positive constants $s_{ij}$ define rates of change of the coupling strengths. This system is coupled by two equations, the first defines behavior of an oscillator, and the second determines dynamics of the coupling strengths. In the special case where $k_{ij}$ is constant and $f_{ij}(\theta_j - \theta_i) = \sin(\theta_j - \theta_i)$, system (1) becomes the Kuramoto model [14]. If $f_{ij}(\theta_j - \theta_i) = \sin(\theta_j - \theta_i)$ and $F_{ij}(\theta_j - \theta_i) = \cos(\theta_j - \theta_i)$, then the system (1)–(2) becomes a generalized Kuramoto model [22]. It is called a homogeneous oscillators system if all intrinsic frequencies of oscillators are equal, i.e. there exists a constant $\omega$ such that $\omega_1 = \omega_2 = \cdots = \omega_n = \omega$. The synchronization of phase-coupled oscillators may appear at two different levels: frequency synchronization and phase synchronization. The synchronization, including frequency synchronization and phase synchronization, has been widely investigated in recent literatures, see [1,4,9–11,15,16,21] and references therein.

Our objective in this work is to develop local phase synchronization and clustering criteria for phase-coupled homogeneous oscillators system with non-constant coupling, distributed delay and general network topology. The distributed delays are incorporated into (1)–(2), so we obtain

$$\dot{\theta}_i(t) = \omega + \sum_{j \in N_i} k_{ij} f\left(\int_{-\tau}^{0} \mu(s)\theta_j(t+s)ds - \theta_i(t)\right), \tag{3}$$

$$\dot{k}_{ij}(t) = s_{ij}\left(\alpha_{ij} F\left(\int_{-\tau}^{0} \mu(s)\theta_j(t+s)ds - \theta_i(t)\right) - k_{ij}\right), \tag{4}$$

where $\tau$ is the maximal coupling delay and the distribution is normalized so that $\int_{-\tau}^0 \mu(s)ds = 1$. Naturally, to specify a solution of (3)–(4), we will need to prescribe the initial conditions

$$\theta_i(t) = \gamma_i(t) \text{ for all } t \in [-\tau, 0], 1 \le i \le n,$$

where each $\gamma_i$ is a continuous function. For more discussions why time delay should be introduced, we refer to [2,5,16,17,19] and references therein.

In what follows, we say that system (3)–(4) achieves *frequency synchronization* if $\lim_{t\to\infty} \dot{\theta}_1(t) = \cdots = \lim_{t\to\infty} \dot{\theta}_n(t) = \Omega$ and $\lim_{t\to\infty} \dot{k}_{ij}(t) = 0$ for all $i, j$, where $\Omega$ is a common synchronization frequency. We say that system (3)–(4) reaches *phase synchronization* if it achieves frequency synchronization, and $\lim_{t\to\infty}(\theta_1(t) - \Omega t) = \lim_{t\to\infty}(\theta_2(t) - \Omega t) = \cdots = \lim_{t\to\infty}(\theta_n(t) - \Omega t) = \varphi$ (a constant). We say that a system reaches $p$ *clusters phase synchronization* if it achieves frequency synchronization and for some set of initial values there exist some constants $\varphi_j \in \mathbb{R}$ and sets $P_j \subset \{1, 2, \cdots, n\}$ satisfying $P_j \cap P_i = \emptyset$ (empty set), $\varphi_j \neq \varphi_i$ (whenever $i \neq j$) and $\cup_j P_j = \{1, 2, \cdots, n\}$, such that $\lim_{t\to\infty}(\theta_i(t) - \Omega t) = \varphi_j$ for all $i \in P_j$, $j = 1, 2, \cdots, p$. The numbers $\varphi_j$ are then called the *clustering phases*.

The structure of the article is as follows. In the next section we linearize the delayed phase-coupled oscillators system and normalize the adjoint matrix. In Section 3, we provide the normal zero-one vectors decomposition in the nullspace. In section 4, a necessary and sufficient condition for clusters phase synchronization is obtained. In this section, both frequency synchronization criteria and phase synchronization criteria are also described. The final synchronization frequency and synchronization phase are formulated in terms of coupling strength, initial values and time delay.