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Regularity results for a class of obstacle problems with nonstandard growth

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A B S T R A C T

We consider the obstacle problem related to the following energy with nonstandard growth

$$\int_\Omega |Du|^{p(x)} \log(e + |Du|)\, dx.$$

We investigate the regularity properties of solutions to the obstacle problems along with a suitable assumptions on the variable exponent $p(\cdot)$ and the obstacle.

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1. Introduction

In this paper, we study the regularity theory for solutions to a certain obstacle problem with nonstandard growth. Let Ω be a bounded open set in $\mathbb{R}^n$ $(n \geq 2)$, $p(\cdot) : \Omega \to [\gamma_1, \gamma_2]$ with $1 < \gamma_1 \leq \gamma_2 < \infty$ be a continuous function, and the functions $\Phi : \Omega \times [0, \infty) \to [0, \infty)$ and $\partial\Phi : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be denoted by

$$\Phi(x, t) := |t|^{p(x)} \log(e + t) \quad \text{and} \quad \partial\Phi(x, \xi) := D_\xi(\Phi(x, |\xi|)), \tag{1.1}$$

where D_ξ is the gradient with respect to the ξ-variable. For a function $\psi : \Omega \to [-\infty, \infty]$ called the *obstacle*, we define a functions space by

$$\mathcal{K}^\Phi_\psi(\Omega) := \left\{ f \in W^{1,\Phi}(\Omega) : f \geq \psi \text{ a.e. in } \Omega \right\}.$$

Here, $W^{1,\Phi}(\Omega)$ is a Sobolev space related to the function Φ, for which we will introduce in the next section. In this setting, we say a function $u \in \mathcal{K}^\Phi_\psi(\Omega)$ is a solution to the *obstacle problem* of $\mathcal{K}^\Phi_\psi(\Omega)$ if it satisfies

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$$\int_\Omega \partial\Phi(x, Du) \cdot D(\varphi - u)\, dx \geq 0$$

for all $\varphi \in \mathcal{K}^\Phi_\psi(\Omega)$ with $\varphi - u$ having a compact support in Ω, which is equivalent to that

$$\int_\Omega \partial\Phi(x, Du) \cdot D\varphi\, dx \geq 0 \tag{1.2}$$

for all $\varphi \in W^{1,\Phi}(\Omega)$ with a compact support and $\varphi \geq \psi - u$ a.e. in Ω. Under the above setting, we will prove the following regularity properties for solutions to the obstacle problem of Φ.

Theorem 1.1. *Suppose the variable exponent $p(\cdot)$ is log-Hölder continuous, that is, $p(\cdot)$ satisfies*

$$L := \sup_{0<r<\frac{1}{2}} \omega(r) \log\left(\frac{1}{r}\right) < \infty, \tag{1.3}$$

where $\omega(\cdot) : [0, \infty) \to [0, \infty)$ is the modulus of continuity of $p(\cdot)$, and the obstacle ψ is Hölder continuous. Let $u \in \mathcal{K}^\Phi_\psi(\Omega)$ be a solution to the obstacle problem of $\mathcal{K}^\Phi_\psi(\Omega)$. Then u is Hölder continuous.

Theorem 1.2. *Suppose the variable exponent $p(\cdot)$ and the gradient of the obstacle ψ are Hölder continuous. Let $u \in \mathcal{K}^\Phi_\psi(\Omega)$ be a solution to the obstacle problem of $\mathcal{K}^\Phi_\psi(\Omega)$. Then Du is Hölder continuous.*

The Obstacle problems are strongly related to many physical phenomena hence the study of those problems is one of main topics in the fields of the calculus of variation and the partial differential equation, see for instance the monograph [31]. Essentially, they are linked to the minimizing problems of energy functionals (for example)

$$\int_\Omega \mathcal{F}(x, Du)\, dx \tag{1.4}$$

in convex admissible sets constrained by obstacle functions. Here, the density function $\mathcal{F} : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfies a suitable convexity condition. Indeed, from a basic computation, see for instance [29], we see that for $g \in W^{1,\Phi}(\Omega)$ with $\psi \leq g$ on $\partial\Omega$, the minimizer of the following energy functional

$$u \in \{w \in \mathcal{K}^\Phi_\psi(\Omega) : w = g \text{ on } \partial\Omega\} \quad \mapsto \quad \int_\Omega \Phi(x, |Du|)\, dx \tag{1.5}$$

is the solution to the obstacle problem of $\mathcal{K}^\Phi_\psi(\Omega)$ with $u = g$ on $\partial\Omega$.

The regularity theory for the elliptic obstacle problems with standard growth, i.e., $\mathcal{F}(x, \xi) \approx |\xi|^p$ with $1 < p < \infty$ in (1.4), is now well understood, for which we refer to classical works [36,7,6] and related references, and its parabolic counterpart has been recently developed, see for instance [4,39,23].

A first relevant extension of such results to the setting of non-uniformly elliptic operators has been obtained in the setting of functionals with $p(x)$-growth, i.e., $\mathcal{F}(x, \xi) \approx |\xi|^{p(x)}$. Regularity problems for minimizers of this functional have been intensively studied in last twenty years; see for instance [1,11] and related references. In particular, for the obstacle problems we refer to [17–20,29,5], where sharp regularity is obtained starting from the techniques developed in the unconstrained case.

Over the recent years, interest in so-called non-autonomous functionals with non-standard growth conditions, has been rapidly increasing. In this situation the functionals with $p(x)$-growth are a particular case. These are indeed functionals as in (1.4) having an energy density $\mathcal{F}$ with both ellipticity or growth

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