J. Math. Anal. Appl. 444 (2016) 980–999

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Scaling invariant Harnack inequalities in a general setting

Wolfhard Hansen [a,*], Ivan Netuka [b]

[a] Fakultät für Mathematik, Universität Bielefeld, 33501 Bielefeld, Germany
[b] Charles University, Faculty of Mathematics and Physics, Mathematical Institute, Sokolovská 83, 186 75 Praha 8, Czech Republic

ARTICLE INFO

Article history:
Received 16 April 2016
Available online 12 July 2016
Submitted by U. Stadtmueller

Keywords:
Harnack inequality
Hölder continuity
Lévy process
Green function
Krylov–Safonov estimate
Ikeda–Watanabe formula

ABSTRACT

In a setting, where only "exit measures" are given, as they are associated with an arbitrary right continuous strong Markov process on a separable metric space, we provide simple criteria for the validity of Harnack inequalities for positive harmonic functions. These inequalities are scaling invariant with respect to a metric on the state space which, having an associated Green function, may be adapted to the special situation. In many cases, this also implies continuity of harmonic functions and Hölder continuity of bounded harmonic functions. The results apply to large classes of Lévy (and similar) processes.

1. Overview

The study of Harnack inequalities for positive functions which are harmonic with respect to rather general partial differential operators of second order, diffusions respectively, has a long history (see [17] and the references therein). Fairly recently, during the last 15 years, Harnack inequalities have been investigated for harmonic functions with respect to various classes of discontinuous Markov processes, integro-differential operators respectively (see [1,2,7,9,14,18–24,26–29]).

The aim of this paper is to offer a very general analytic approach to scaling invariant Harnack inequalities for positive universally measurable functions on a separable metric space X which are harmonic with respect to given "harmonic measures" μ_x^U (not charging U), $x \in U$, U open in X (see (2.6)). For $V \subset U$ the corresponding measures are supposed to be compatible in a way which is obvious for exit distributions of right continuous strong Markov processes and harmonic measures on balayage spaces (see Examples 2.1). An additional ingredient we shall need is a "quasi-capacity" on X having suitable scaling properties such that an estimate of Krylov–Safonov type holds (see (3.3)).

Then a certain property (HJ) of the measures μ_x^U, which in Examples 2.1 trivially holds for diffusions and harmonic spaces, is necessary and sufficient for the validity of scaling invariant Harnack inequalities

* Corresponding author.
E-mail addresses: hansen@math.uni-bielefeld.de (W. Hansen), netuka@karlin.mff.cuni.cz (I. Netuka).

http://dx.doi.org/10.1016/j.jmaa.2016.06.069
0022-247X/

(Theorem 3.3). For Lévy processes it is easy to specify simple properties of the Lévy measure which imply (HJ) for the exit distributions (see, for example, Lemma 5.1).

In Section 4, we discuss properties of an associated "Green function" which allow us to prove a Krylov–Safonov estimate for the corresponding capacity. This leads to Theorem 4.10 and, using recent results on Hölder continuity from [12], to Theorem 4.12 on (Hölder) continuity of harmonic functions.

After a first application to Lévy processes (Theorem 5.2) we discuss consequences of an estimate of Ikeda–Watanabe type (Theorems 6.2 and 6.3).

In a final Section 7, we indicate how a Green function satisfying (only) a weak 3G-property leads to Harnack inequalities which are scaling invariant with respect to an intrinsically defined metric.

2. Harmonic measures and harmonic functions

Let (X, ρ) be a separable metric space. In fact, the separability will only be used to ensure that finite measures μ on its σ-algebra $\mathcal{B}(X)$ of Borel subsets satisfy

$$\mu(A) = \sup\{\mu(F)\colon F \text{ closed}, F \subset A\}, \qquad A \in \mathcal{B}(X) \tag{2.1}$$

(recall that every finite measure on the completion of X is tight).

For every open set Y in X, let $\mathcal{U}(Y)$ denote the set of all open sets U such that the closure $\overline{U}$ of U is contained in Y. Given a set $\mathcal{F}$ of numerical functions on X, let $\mathcal{F}_b$, $\mathcal{F}^+$ be the set of all functions in $\mathcal{F}$ which are bounded, positive respectively. Let $\mathcal{M}(X)$ denote the set of all finite measures on $(X, \mathcal{B}(X))$ (which we also consider as measures on the σ-algebra $\mathcal{B}^*(X)$ of all universally measurable sets). For every $\mu \in \mathcal{M}(X)$, let $\|\mu\|$ denote the total mass $\mu(X)$.

For sufficient flexibility in applications, we consider harmonic measures only for open sets which are contained in a given open set X_0 of X. More precisely, we suppose that we have measures $\mu_x^U \in \mathcal{M}(X)$, $x \in X$, $U \in \mathcal{U}(X_0)$, such that the following hold for all $x \in X$ and $U, V \in \mathcal{U}(X_0)$ (where ε_x is the Dirac measure at x):

(M_0) *The measure μ_x^U is supported by U^c, and $\|\mu_x^U\| \le 1$. If $x \in U^c$, then $\mu_x^U = \varepsilon_x$.*
(M_1) *The functions $y \mapsto \mu_y^U(E)$, $E \in \mathcal{B}(X)$, are universally measurable on X and*

$$\mu_x^U = (\mu_x^V)^U := \int \mu_y^U \, d\mu_x^V(y), \qquad \text{if } V \subset U. \tag{2.2}$$

Of course, stochastic processes and potential theory abundantly provide examples (with $X_0 = X$).

Examples 2.1. 1. Right process $\mathfrak{X}$ with strong Markov property on a Radon space X and

$$\mu_x^U(E) := \mathbb{P}^x[X_{\tau_U} \in E], \qquad E \in \mathcal{B}(X),$$

where $\tau_U := \inf\{t \ge 0\colon X_t \in U^c\}$ ([4, Propositions 1.6.5 and 1.7.11, Theorem 1.8.5]).

If $U, V \in \mathcal{U}(X_0)$ with $V \subset U$, then $\tau_U = \tau_V + \tau_U \circ \theta_{\tau_V}$, and hence, by the strong Markov property, for all $x \in X$ and $E \in \mathcal{B}(X)$,

$$\mu_x^U(E) = \mathbb{P}^x[X_{\tau_U} \in E] = \mathbb{E}^x\left(\mathbb{P}^{X_{\tau_V}}[X_{\tau_U} \in E]\right) = \int \mu_y^U(E)\, d\mu_x^V(y).$$

2. Balayage space $(X, \mathcal{W})$ (see [5,10]) such that $1 \in \mathcal{W}$,

$$\int v \, d\mu_x^U = R_v^{U^c}(x) := \inf\{w(x)\colon w \in \mathcal{W},\ w \ge v \text{ on } U^c\}, \qquad v \in \mathcal{W}.$$