J. Math. Anal. Appl. 444 (2016) 1000–1014

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Multiple-angle formulas of generalized trigonometric functions with two parameters ☆

Shingo Takeuchi

Department of Mathematical Sciences, Shibaura Institute of Technology, Japan [1]

ARTICLE INFO

Article history:
Received 19 April 2016
Available online 11 July 2016
Submitted by S. Cooper

Dedicated to Professor Eiji Yanagida on the occasion of his 60th birthday

Keywords:
Multiple-angle formulas
Generalized trigonometric functions
p-Laplacian
Eigenvalue problems
Pendulum equation
Lemniscate

ABSTRACT

Generalized trigonometric functions with two parameters were introduced by Drábek and Manásevich to study an inhomogeneous eigenvalue problem of the p-Laplacian. Concerning these functions, no multiple-angle formula has been known except for the classical cases and a special case discovered by Edmunds, Gurka and Lang, not to mention addition theorems. In this paper, we will present new multiple-angle formulas which are established between two kinds of the generalized trigonometric functions, and apply the formulas to generalize classical topics related to the trigonometric functions and the lemniscate function.
© 2016 Elsevier Inc. All rights reserved.

1. Introduction

Let $p,\ q \in (1,\infty)$ be any constants. We define $\sin_{p,q} x$ by the inverse function of

$$\sin_{p,q}^{-1} x := \int_0^x \frac{dt}{(1-t^q)^{1/p}}, \quad 0 \le x \le 1,$$

and

$$\pi_{p,q} := 2\sin_{p,q}^{-1} 1 = 2\int_0^1 \frac{dt}{(1-t^q)^{1/p}} = \frac{2}{q}B\left(\frac{1}{p^*},\frac{1}{q}\right), \tag{1.1}$$

☆ This work was supported by JSPS KAKENHI Grant Number 24540218.
E-mail address: shingo@shibaura-it.ac.jp.

[1] 307 Fukasaku, Minuma-ku, Saitama-shi, Saitama 337-8570, Japan.

http://dx.doi.org/10.1016/j.jmaa.2016.06.074
0022-247X/© 2016 Elsevier Inc. All rights reserved.

where $p^* := p/(p-1)$ and B denotes the beta function. The function $\sin_{p,q} x$ is increasing in $[0, \pi_{p,q}/2]$ onto $[0, 1]$. We extend it to $(\pi_{p,q}/2, \pi_{p,q}]$ by $\sin_{p,q}(\pi_{p,q} - x)$ and to the whole real line $\mathbb{R}$ as the odd $2\pi_{p,q}$-periodic continuation of the function. Since $\sin_{p,q} x \in C^1(\mathbb{R})$, we also define $\cos_{p,q} x$ by $\cos_{p,q} x := (\sin_{p,q} x)'$. Then, it follows that

$$|\cos_{p,q} x|^p + |\sin_{p,q} x|^q = 1.$$

In case $p = q = 2$, it is obvious that $\sin_{p,q} x$, $\cos_{p,q} x$ and $\pi_{p,q}$ are reduced to the ordinary $\sin x$, $\cos x$ and π, respectively. This is a reason why these functions and the constant are called *generalized trigonometric functions* (with parameter (p, q)) and *the generalized* π, respectively.

Drábek and Manásevich [5] introduced the generalized trigonometric functions with two parameters to study an inhomogeneous eigenvalue problem of p-Laplacian. They gave a closed form of solutions (λ, u) of the eigenvalue problem

$$-(|u'|^{p-2}u')' = \lambda|u|^{q-2}u, \quad u(0) = u(L) = 0.$$

Indeed, for any $n = 1, 2, \ldots$, there exists a curve of solutions $(\lambda_{n,R}, u_{n,R})$ with a parameter $R \in \mathbb{R} \setminus \{0\}$ such that

$$\lambda_{n,R} = \frac{q}{p^*}\left(\frac{n\pi_{p,q}}{L}\right)^p |R|^{p-q}, \tag{1.2}$$

$$u_{n,R}(x) = R\sin_{p,q}\left(\frac{n\pi_{p,q}}{L}x\right) \tag{1.3}$$

(see also [12]). Conversely, there exists no other solution of the eigenvalue problem. Thus, the generalized trigonometric functions play important roles to study problems of the p-Laplacian.

It is of interest to know whether the generalized trigonometric functions have multiple-angle formulas unless $p = q = 2$. A few multiple-angle formulas seem to be known. Actually, in case $2p = q = 4$, the function $\sin_{p,q} x = \sin_{2,4} x$ coincides with the lemniscate sine function $\mathrm{sl}\, x$, whose inverse function is defined as

$$\mathrm{sl}^{-1} x := \int_0^x \frac{dt}{\sqrt{1-t^4}}.$$

Furthermore, $\pi_{2,4}$ is equal to the lemniscate constant $\varpi := 2\,\mathrm{sl}^{-1} 1 = 2.6220\cdots$. Concerning $\mathrm{sl}\, x$ and ϖ, we refer to the reader to [11, p. 81], [15] and [16, §22.8]. Since $\mathrm{sl}\, x$ has the multiple-angle formula

$$\mathrm{sl}\,(2x) = \frac{2\,\mathrm{sl}\, x\sqrt{1-\mathrm{sl}^4 x}}{1+\mathrm{sl}^4 x}, \quad 0 \le x \le \frac{\varpi}{2}, \tag{1.4}$$

we see that

$$\sin_{2,4}(2x) = \frac{2\sin_{2,4} x \cos_{2,4} x}{1+\sin_{2,4}^4 x}, \quad 0 \le x \le \frac{\pi_{2,4}}{2}.$$

Also in case $p^* = q = 4$, it is possible to show that $\sin_{p,q} x = \sin_{4/3,4} x$ can be expressed in terms of the Jacobian elliptic function, whose multiple-angle formula yields

$$\sin_{4/3,4}(2x) = \frac{2\sin_{4/3,4} x \cos_{4/3,4}^{1/3} x}{\sqrt{1+4\sin_{4/3,4}^4 x \cos_{4/3,4}^{4/3} x}} \quad 0 \le x < \frac{\pi_{4/3,4}}{4}. \tag{1.5}$$

Download English Version:

https://daneshyari.com/en/article/4614192

Download Persian Version:

https://daneshyari.com/article/4614192

Daneshyari.com