J. Math. Anal. Appl. 444 (2016) 1133–1154

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Rademacher functions in Morrey spaces ☆

Sergei V. Astashkin [a,b,*], Lech Maligranda [c]

[a] Department of Mathematics and Mechanics, Samara State University, Acad. Pavlova 1, 443011 Samara, Russia
[b] Samara State Aerospace University (SSAU), Moskovskoye shosse 34, 443086, Samara, Russia
[c] Department of Mathematics, Luleå University of Technology, SE-971 87 Luleå, Sweden

ARTICLE INFO

Article history:
Received 20 January 2016
Available online 14 July 2016
Submitted by P.G. Lemarie-Rieusset

Keywords:
Rademacher functions
Morrey spaces
Korenblyum–Krein–Levin spaces
Marcinkiewicz spaces
Complemented subspaces

ABSTRACT

The Rademacher sums are investigated in the Morrey spaces $M_{p,w}$ on $[0,1]$ for $1 \le p < \infty$ and weight w being a quasi-concave function. They span l_2 space in $M_{p,w}$ if and only if the weight w is smaller than $\log_2^{-1/2} \frac{2}{t}$ on $(0,1)$. Moreover, if $1 < p < \infty$ the Rademacher subspace $\mathcal{R}_{p,w}$ is complemented in $M_{p,w}$ if and only if it is isomorphic to l_2. However, the Rademacher subspace $\mathcal{R}_{1,w}$ is not complemented in $M_{1,w}$ for any quasi-concave weight w. In the last part of the paper geometric structure of Rademacher subspaces in Morrey spaces $M_{p,w}$ is described. It turns out that for any infinite-dimensional subspace X of $\mathcal{R}_{p,w}$ the following alternative holds: either X is isomorphic to l_2 or X contains a subspace which is isomorphic to c_0 and is complemented in $\mathcal{R}_{p,w}$.

1. Introduction and preliminaries

The well-known Morrey spaces introduced by Morrey in 1938 [20] in relation to the study of partial differential equations were widely investigated during last decades, including the study of classical operators of harmonic analysis: maximal, singular and potential operators—in various generalizations of these spaces. In the theory of partial differential equations, along with the weighted Lebesgue spaces, Morrey-type spaces also play an important role. They appeared to be quite useful in the study of the local behavior of the solutions of partial differential equations, a priori estimates and other topics.

Let $0 < p < \infty$, w be a non-negative non-decreasing function on $[0, \infty)$, and Ω a domain in $\mathbb{R}^n$. The *Morrey space* $M_{p,w} = M_{p,w}(\Omega)$ is the class of Lebesgue measurable real functions f on Ω such that

☆ Research of the first named author was partially supported by the Ministry of Education and Science of the Russian Federation.
* Corresponding author.
E-mail addresses: astash@samsu.ru (S.V. Astashkin), lech.maligranda@ltu.se (L. Maligranda).

http://dx.doi.org/10.1016/j.jmaa.2016.07.008
0022-247X/

$$\|f\|_{M_{p,w}} = \sup_{0<r<\mathrm{diam}(\Omega),\, x_0\in\Omega} w(r) \left(\frac{1}{|B_r(x_0)|} \int\limits_{B_r(x_0)\cap\Omega} |f(t)|^p \, dt \right)^{1/p} < \infty, \tag{1}$$

where $B_r(x_0)$ is a ball with the center at x_0 and radius r. It is a quasi-Banach ideal space on Ω. The so-called ideal property means that if $|f| \le |g|$ a.e. on Ω and $g \in M_{p,w}$, then $f \in M_{p,w}$ and $\|f\|_{M_{p,w}} \le \|g\|_{M_{p,w}}$. In particular, if $w(r) = 1$ then $M_{p,w}(\Omega) = L_\infty(\Omega)$, if $w(r) = r^{1/p}$ then $M_{p,w}(\Omega) = L_p(\Omega)$ and in the case when $w(r) = r^{1/q}$ with $0 < p \le q < \infty$ $M_{p,w}(\Omega)$ are the classical Morrey spaces, denoted shortly by $M_{p,q}(\Omega)$ (see [14, Part 4.3], [15,23] and [29]). Moreover, as a consequence of the Hölder–Rogers inequality we obtain monotonicity with respect to p, that is,

$$M_{p_1,w}(\Omega) \overset{1}{\hookrightarrow} M_{p_0,w}(\Omega) \quad \text{if} \quad 0 < p_0 \le p_1 < \infty.$$

For two quasi-Banach spaces X and Y the symbol $X \overset{C}{\hookrightarrow} Y$ means that the embedding $X \subset Y$ is continuous and $\|f\|_Y \le C\|f\|_X$ for all $f \in X$.

It is easy to see that in the case when $\Omega = [0, 1]$ quasi-norm (1) can be defined as follows

$$\|f\|_{M_{p,w}} = \sup_I \; w(|I|) \left(\frac{1}{|I|} \int\limits_I |f(t)|^p \, dt \right)^{1/p}, \tag{2}$$

where the supremum is taken over all intervals I in $[0, 1]$. In what follows $|E|$ is the Lebesgue measure of a set $E \subset \mathbb{R}$.

The main purpose of this paper is the investigation of the behavior of Rademacher sums

$$R_n(t) = \sum_{k=1}^n a_k r_k(t), \;\; a_k \in \mathbb{R} \text{ for } k = 1, 2, ..., n, \;\text{ and } n \in \mathbb{N}$$

in general Morrey spaces $M_{p,w}$. Recall that the Rademacher functions on $[0, 1]$ are defined by $r_k(t) = \mathrm{sign}(\sin 2^k \pi t)$, $k \in \mathbb{N}, t \in [0, 1]$.

By $\mathcal{R}_{p,w}$ we denote the subspace spanned by the Rademacher functions r_k, $k = 1, 2, \ldots$ in $M_{p,w}$.

The most important tool in studying Rademacher sums in the classical L_p-spaces and in general rearrangement invariant spaces is the so-called *Khintchine inequality* (cf. [11, p. 10], [1, p. 133], [16, p. 66] and [4, p. 743]): if $0 < p < \infty$, then there exist constants $A_p, B_p > 0$ such that for any sequence of real numbers $\{a_k\}_{k=1}^n$ and any $n \in \mathbb{N}$ we have

$$A_p \Big(\sum_{k=1}^n |a_k|^2 \Big)^{1/2} \le \|R_n\|_{L_p[0,1]} \le B_p \Big(\sum_{k=1}^n |a_k|^2 \Big)^{1/2}. \tag{3}$$

Therefore, for any $1 \le p < \infty$, the Rademacher functions span in L_p an isomorphic copy of l_2. Also, the subspace $[r_n]$ is complemented in L_p for $1 < p < \infty$ and is not complemented in L_1 since no complemented infinite dimensional subspace of L_1 can be reflexive. In L_∞, the Rademacher functions span an isometric copy of l_1, which is uncomplemented.

The only non-trivial estimate for Rademacher sums in a general rearrangement invariant (r.i.) space X on $[0, 1]$ is the inequality

$$\|R_n\|_X \le C \Big(\sum_{k=1}^n |a_k|^2 \Big)^{1/2}, \tag{4}$$