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Weak asymptotic methods for scalar equations and systems

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ABSTRACT

In this paper we show how one can construct families of continuous functions which satisfy asymptotically scalar equations with discontinuous nonlinearity and systems having irregular solutions. This construction produces weak asymptotic methods which are issued from Maslov asymptotic analysis. We obtain a sequence of functions which tend to satisfy the equation(s) in the weak sense in the space variable and in the strong sense in the time variable. To this end we reduce the partial differential equations to a family of ordinary differential equations in a classical Banach space. For scalar equations we prove that the initial value problem is well posed in the L^1 sense for the approximate solutions we construct. Then we prove that this method gives back the widely accepted solutions when they are known. For systems we obtain existence in the general case and uniqueness in the analytic case using an abstract Cauchy–Kovalevska theorem.

1. Introduction

We construct sequences of continuous functions that tend to satisfy asymptotically scalar equations and systems in presence of irregular data and/or solutions when a small parameter tends to 0. More precisely, we show that these sequences tend to satisfy the equation(s) in a weak sense in the space variables and in the strong sense in the time variable. A weak asymptotic solution for the scalar equation

$$\partial_t u(x,t) + \sum_{i=1}^{n} \partial_{x_i}[u(x,t) f_i(u(x,t),x,t)] = 0, \ \ u(x,0) = u_0(x) \tag{1}$$

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in n-space dimension where $x \in \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, the n-D torus, $t \in [0, +\infty[$, $f_i : \mathbb{R} \times \mathbb{T}^n \times [0, +\infty[\longmapsto \mathbb{R}$, $u : \mathbb{T}^n \times [0, +\infty[\longmapsto \mathbb{R}$, is a sequence $(u_\epsilon)_\epsilon$ of functions of class $\mathcal{C}^1$ in t and of class $\mathcal{C}^0$ or L^∞ in x such that $\forall \psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ $\forall t$

$$\int_{\mathbb{R}^n} [\partial_t u(x,t,\epsilon)\psi(x) - \sum_{i=1}^{n} u(x,t,\epsilon) f_i(u(x,t,\epsilon),x,t)\partial_{x_i}\psi(x)]dx \to 0 \tag{2}$$

when $\epsilon \to 0$, plus the initial condition. The nonlinearities f_i can be discontinuous in the x variable but are regular in u and t. Then we consider systems of the form

$$\partial_t u_j + \sum_{i=1}^{n} \partial_{x_i}(u_j f_{i,j}(u_1, \dots, u_p, x, t)) = \sum_{i=1}^{n} \partial_{x_i}(A_{i,j}(u_1, \dots, u_p, x, t)), \tag{3}$$

$1 \le j \le p$ and with the same assumption on the nonlinearities $f_{i,j}$ and $A_{i,j}$. We show in this paper that the results we obtain always coincide with the well known and widely accepted solutions in the scalar case, and, when the data are analytic, in the case of systems, using an abstract Cauchy Kovalevska theorem. In particular for important systems of physics close to (3), such as the selfgravitating pressureless fluids, the isothermal and isentropic gases, the shallow water equations and the standard system of two fluid flows [28,29,31], verifications that the solutions we obtain coincide with the known solutions have been done numerically.

Weak asymptotic methods [34,38,40,65,66] have been introduced by V. Danilov et al. in the framework of the Maslov–Whitham asymptotic analysis. They have proved to be an efficient mathematical tool to study creation and superposition of singular solutions to various nonlinear PDEs, such as δ-waves and the more general $\delta^{(n)}$-waves, in particular the trend of works by S. Albeverio, V. Danilov, V. Shelkovich et al. [3–5,33–39,65–67]. Similar approximate solutions with full proofs had already been obtained in the case of spherical symmetry by various authors, in particular Joseph et al. [23,45].

The weak asymptotic methods presented in this paper are constructed by transforming each scalar PDE into a family of ODEs of the Lipschitz type in the Banach space $\mathcal{C}(\mathbb{T}^n)$ or $L^\infty(\mathbb{T}^n)$ (of continuous and essentially bounded functions respectively).

This work is a direct continuation of previous results for the 3-D selfgravitating pressureless fluids [28], for the 3-D isothermal and isentropic gases and the 2-D shallow water equations [29] and the 1-D standard model of two fluid flows [31]. The method to reduce the partial differential equations to a family of ordinary differential equations is constructive, which has permitted to verify the results from simple numerical schemes for ODEs, such as the Euler and Runge–Kutta methods, in the case of previously known solutions. In the absence of known solutions the proof that the solutions of the ODEs tend to satisfy the equations and the fact that one can use classical convergent numerical schemes for ODEs give confidence in the results so obtained.

A motivation is the occurrence of models such as (1) in numerous engineering problems. There are a number of prototype relevant models with discontinuous nonlinearities in *oil trapping phenomenon* [7,8,21,46,61,69], *a Whitham model of car traffic flow on a highway* [44,59] and a model of *continuous sedimentation in ideal clarifier-thickener units* [19], see also [15,48]. Scalar equations and systems with discontinuous nonlinearities also arise in sedimentation processes [41], in radar shape-from-shading problems [63] and as well as building blocks in numerical methods for Hamilton–Jacobi equations [47]. Equations of the form (1) with discontinuous in x nonlinearity f attracted much attention in the recent past, because of the difficulties of adaptation of the approach developed for the smooth case, due in part to the presence of several different stable solutions with same initial data. In the context of Buckley–Leverett equations, each notion of solution is uniquely determined by the choice of a connection, which is made unique at the interface by a proper choice of a stable solution from many classes of stable solutions, [7] and [11]. Nonclassical