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Summable solutions of some partial differential equations and generalised integral means

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ABSTRACT

We describe partial differential operators for which we can construct generalised integral means satisfying Pizzetti-type formulas. Using these formulas we give a new characterisation of summability of formal power series solutions to some multidimensional partial differential equations in terms of holomorphic properties of generalised integral means of the Cauchy data.

1. Introduction

We consider the initial value problem for a multidimensional linear partial differential equation with constant coefficients

$$(\partial_t - P(\partial_z))u = 0, \qquad u(0,z) = \varphi(z), \tag{1}$$

where $t \in \mathbb{C}$, $z \in \mathbb{C}^n$, $P(\partial_z) \in \mathbb{C}[\partial_z]$ is a differential operator of order greater than 1 with complex coefficients and φ is holomorphic in a complex neighbourhood of the origin. The unique formal power series solution $\widehat{u}$ of (1) is in general divergent. So, it is natural to ask about sufficient and necessary conditions (expressed in terms of the Cauchy datum φ) under which the formal solution $\widehat{u}$ is convergent or, more generally, summable.

If the spatial variable z is one-dimensional then, by the general theory introduced by Balser [2], one can find the sufficient condition for summability of $\widehat{u}$ in terms of the analytic continuation property of the Cauchy datum φ (see [2, Theorem 5]). Moreover, in this case, due to a certain symmetry between the variables t and z, this sufficient condition is also necessary (see [10, Theorem 4] and [12]).

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But if the spatial variable z is multidimensional then the characterisation of analytic and summable solutions of (1) in terms of φ is much more complicated.

Such characterisation were found in the special case $P(\partial_z) = \Delta_z = \sum_{k=1}^n \partial_{z_k}^2$, when (1) is the Cauchy problem for the multidimensional heat equation. Namely, in this case Łysik [9] proved that $\widehat{u}$ is convergent if and only if the integral mean of φ over the closed ball $B(x,r)$ or the sphere $S(x,r)$, as a function of the radius r, extends to an entire function of exponential order at most 2. Moreover, the author [11] showed that $\widehat{u}$ is 1-summable in a direction d if and only if the integral mean of φ over the closed ball $B(x,r)$ or the sphere $S(x,r)$, can be analytically continued to infinity in some sectors bisected by $d/2$ and $\pi + d/2$ with respect to r, and this continuation is of exponential order at most 2 as r tends to infinity.

These characterisations are based on the Pizzetti formulas which give the expansions of the integral means of φ in terms of the radius r with coefficients depending on the iterated Laplacians of φ (see Proposition 7).

Such expansion for the surface integral mean in $\mathbb{R}^3$ was established already in 1909 by Pizzetti [15]. His result was extended to $\mathbb{R}^n$ by Nicolesco [13]. Another proof of classical Pizzetti formula in $\mathbb{R}^n$, which is based on so called generating functions of the solid and spherical means, was given recently by Toma [17].

The Pizzetti formula has been also generalised by Gray and Willmore [7] to arbitrary Riemannian manifolds, by Da Lio and Rodino [6] to the case of the heat operator, and by Bonfiglioli [4,5] to the case of some subelliptic operators such as the Kohn Laplace operator on the Heisenberg group and its generalisation to orthogonal sub-Laplacians on H-type groups.

In the paper we are concerned with Pizzetti-type formulas which express a given generalised integral mean in $\mathbb{R}^n$ (or $\mathbb{C}^n$) as a series in the radius r whose coefficients depend on the iterated operator $P(\partial_z)$, where $P(\partial_z)$ is a given partial differential operator with constant coefficients. Observe that our approach to the generalisation of the Pizzetti formula is different from [6] and [4,5], where the coefficients of the expansions are not given directly by the iterated heat operator or by the iterated subelliptic operators.

Generalised integral means and Pizzetti-type formulas are the main tools used in the paper. Using them, we extend the results [9] and [11] to more general multidimensional partial differential operators $P(\partial_z)$.

We show that if a generalised integral mean of φ satisfies a Pizzetti-type formula for the operator $P(\partial_z)$ then we are able to characterise convergent (and under some additional conditions also summable) solutions of (1) in terms of the generalised integral means (Theorem 1). For this reason it is important to describe such operators $P(\partial_z)$, for which we can find generalised integral means satisfying a Pizzetti-type formula.

In the paper we construct such generalised integral means for every homogeneous operator with real coefficients of order 1 (Proposition 6) and for every elliptic homogeneous operator of order 2 with real coefficients (Proposition 9). As a corollary we obtain a characterisation of analytic and 1-summable solutions of (1) in terms of generalised integral means in the case when $P(\partial_z) = \sum_{i,j=1}^n a_{ij}\partial_{z_i z_j}^2$ is a homogeneous elliptic operator of order 2 with real coefficients (Theorem 2). We also prove that it is impossible to find a generalised integral mean satisfying a Pizzetti-type formula for any homogeneous (Theorem 3) or quasi-homogeneous (Theorem 4) operator of order $p > 2$.

We also extend the notion of generalised integral means to the complex case. Then we are able to construct complex generalised integral means satisfying Pizzetti-type formulas for $P(\partial_z) = Q^s(\partial_z)$, where $Q(\partial_z)$ is a homogeneous operator of order $p \leq 2$ and $s \in \mathbb{N}$. In particular, as corollaries we obtain characterisations of analytic and summable solutions of (1) in terms of complex generalised integral means in the case when $P(\partial_z) = \partial_{z_1}^2 - \sum_{k=2}^n \partial_{z_k}^2 = \Box_z$ is the wave operator (Corollary 2), $P(\partial_z) = \Delta_z^s$ is the s-Laplace operator (Corollary 3), $P(\partial_z) = (\sum_{k=1}^n a_k \partial_{z_k})^s$ (Corollary 4) and $P(\partial_z) = (\sum_{i,j=1}^n a_{ij}\partial_{z_i z_j}^2)^s$ (Corollary 5).

2. Notation

Throughout the paper $B(x,r)$ ($S(x,r)$, respectively) denotes the real closed ball (sphere, respectively) with centre at $x \in \mathbb{R}^n$ and radius $r > 0$. Moreover, the complex disc in $\mathbb{C}^n$ with centre at the origin and