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Comparison theorems for conjoined bases of linear Hamiltonian differential systems and the comparative index

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ABSTRACT

In this paper we present comparison results for focal points of conjoined bases $Y(t)$, $\hat{Y}(t)$ of two linear Hamiltonian differential systems under the majorant condition $\mathcal{H}(t) - \hat{\mathcal{H}}(t) \geq 0$ for their Hamiltonians $\mathcal{H}(t)$, $\hat{\mathcal{H}}(t)$. Both systems are considered without controllability (or normality) assumptions and under the Legendre condition for $\hat{\mathcal{H}}(t)$. The main result of the paper connects the difference between the number of proper focal points of $Y(t)$, $\hat{Y}(t)$ with the number of proper focal points of the transformed conjoined basis $\hat{Z}^{-1}(t)Y(t)$, where $\hat{Z}(t)$ is a symplectic fundamental solution matrix of the Hamiltonian system associated with $\hat{\mathcal{H}}(t)$. Focal points of this transformed basis coincide with focal points of the matrix Wronskian of $Y(t)$, $\hat{Y}(t)$. The main tool of the paper is the comparative index theory for discrete symplectic systems generalized to the continuous case.

1. Introduction

Oscillation theory of self-adjoint linear differential equations or systems is a classical research topic [16, 23]. In this paper, we contribute to this theory by considering the linear Hamiltonian systems

$$y'(t) = J\mathcal{H}(t)y(t),\ \mathcal{H}(t) = \begin{bmatrix} -C(t) & A^T(t) \\ A(t) & B(t) \end{bmatrix},\ \mathcal{H}(t) = \mathcal{H}^T(t),\ J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \tag{1.1}$$

and

$$\hat{y}'(t) = J\hat{\mathcal{H}}(t)\hat{y}(t),\ \hat{\mathcal{H}}(t) = \begin{bmatrix} -\hat{C}(t) & \hat{A}^T(t) \\ \hat{A}(t) & \hat{B}(t) \end{bmatrix},\ \hat{\mathcal{H}}(t) = \hat{\mathcal{H}}^T(t),\ t \in [a, b] \tag{1.2}$$

with the piecewise continuous blocks $A(t)$, $B(t)$, $C(t)$, $\hat{A}(t)$, $\hat{B}(t)$, $\hat{C}(t) : [a, b] \to \mathbb{R}^{n\times n}$ under the majorant condition

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$$\mathcal{H}(t) - \hat{\mathcal{H}}(t) \geq 0, \tag{1.3}$$

where $A \geq 0$ means that the symmetric matrix A is nonnegative definite, and I, 0 denote the identity and zero matrices of appropriate dimensions. System (1.2) is considered under the so-called Legendre condition

$$\hat{B}(t) \geq 0,\ t \in [a, b], \tag{1.4}$$

then (1.4), (1.3) imply that the Legendre condition holds for (1.1) as well, i.e. $B(t) \geq 0$, $t \in [a, b]$.

In this paper, we introduce into the consideration a transformed system associated with (1.1), (1.2). Recall that any system in form (1.1) possesses $2n \times n$ matrix solutions $Y(t) = \binom{X(t)}{U(t)}$ (the so-called *conjoined bases*, see [16]) such that

$$X^T(t)U(t) = U^T(t)X(t), \quad \operatorname{rank} Y(t) = n \tag{1.5}$$

and symplectic fundamental solution matrices, i.e. $Z(t) \in \mathbb{R}^{2n\times 2n}$ such that $Z(t)^T J Z(t) = J$. Let $\hat{Z}(t) \in C_p^1$ (i.e., $\hat{Z}(t)$ is continuous with piecewise continuous $\hat{Z}'(t)$) be a symplectic fundamental matrix of system (1.2) and $Y(t) \in C_p^1$ be a conjoined basis of (1.1), then the matrix $\tilde{Y}(t) = \hat{Z}^{-1}(t)Y(t) \in C_p^1$ is the conjoined basis of the transformed Hamiltonian system [4]

$$\tilde{y}'(t) = J\tilde{\mathcal{H}}(t)\tilde{y}(t), \quad \tilde{\mathcal{H}}(t) = \hat{Z}^T(t)(\mathcal{H}(t) - \hat{\mathcal{H}}(t))\hat{Z}(t) = \tilde{\mathcal{H}}^T(t). \tag{1.6}$$

By majorant condition (1.3), the Hamiltonian $\tilde{\mathcal{H}}(t)$ of (1.6) is nonnegative definite, i.e. $\tilde{\mathcal{H}}(t) \geq 0$.

Note also that the upper block $\tilde{X}(t)$ of $\tilde{Y}(t)$ can be presented in terms of the Wronskian of the conjoined bases $Y(t)$ and $\hat{Y}(t) = \hat{Z}(t)[0\, I]^T$, i.e. $\tilde{X}(t) = -\hat{Y}^T(t)JY(t)$. From this point of view the main result of this paper is related to the *renormalized* version of the oscillation theory for differential Hamiltonian systems. For the second order Sturm–Liouville differential equations (which are the special case of (1.1)) the renormalized and more general *relative* oscillation theory is established in [10,14].

In this paper, we are concerned with comparison theorems for systems (1.1), (1.2) incorporating oscillation behavior of system (1.6). In the classical theory, such as in [11], systems (1.1), (1.2) are studied under controllability (or normality) assumption, see [11, Section 4.1]. Controllability means that the solutions $y(t) = \binom{x(t)}{u(t)}$ of (1.1) are not "degenerate" in the first component, that is, whenever $x(t) = 0$ on a subinterval of $[a, b]$, then also $u(t) = 0$ in this subinterval. By [11, Theorem 4.1.3], condition $B(t) \geq 0$ and the controllability assumption yield that the focal points of conjoined bases $Y(t) = \binom{X(t)}{U(t)}$ of system (1.1), i.e., the points $t_0 \in [a, b]$ at which $X(t_0)$ is singular, are isolated. The multiplicity of such a focal point is then the dimension of the kernel of $X(t_0)$, i.e., $\operatorname{def} X(t_0)$. The most general comparison result [5, Proposition 1], [11, Theorem 7.3.1] presents estimates for the difference between the number of focal points of conjoined bases $Y(t)$ and $\hat{Y}(t) = \binom{\hat{X}(t)}{\hat{U}(t)}$ of (1.1) and (1.2) in (a, b) in terms of $\operatorname{ind}(Q - \hat{Q})(t\pm)$, $t = a, b$, where $Q(t) = U(t)X^{-1}(t)$, $\hat{Q}(t) = \hat{U}(t)\hat{X}^{-1}(t)$ and *ind* denotes the index, i.e. the number of negative eigenvalues. Note that [11, Theorem 7.3.1] covers classical results concerning inequalities for solutions of the Riccati equations [15] associated with (1.1), (1.2).

In [12] and followed by [22], the concept of possibly "abnormal" linear Hamiltonian systems was introduced. Based on the results of [12, Theorem 3] saying that $B(t) \geq 0$ implies a piecewise constant kernel of $X(t)$ on $[a, b]$, a new notion of *proper focal points* was given in [22]. First Sturmian-type results for differential Hamiltonian systems without normality and their applications are presented in [2,17–19].

The main result of this paper (Theorem 2.2) connects the multiplicities of proper focal points of conjoined bases of (1.1), (1.2), and (1.6) for the abnormal case replacing the quantities $\operatorname{ind}(\hat{Q} - Q)(t\pm)$, $t = a$, b in [11, Theorem 7.3.1] by the so-called *comparative index* for $Y(t)$, $\hat{Y}(t)$ at the endpoints $t = a$ and $t = b$. This comparative index notion [6,7] was introduced and elaborated for discrete symplectic systems [1] which