J. Math. Anal. Appl. 444 (2016) 1304–1320

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Explicit formulas for the distribution of complex zeros of a family of random sums

Andrew Ledoan

Department of Mathematics, University of Tennessee at Chattanooga, 415 EMCS Building (Dept. 6956), 615 McCallie Avenue, Chattanooga, TN 37403-2598, United States

ARTICLE INFO

Article history:
Received 30 May 2016
Available online 19 July 2016
Submitted by B.C. Berndt

Keywords:
Cholesky factorization
Expected number of zeros
Explicit formula
Random sum

ABSTRACT

The present paper provides an explicit formula for the average intensity of the distribution of complex zeros of a family of random sums of the form $S_n(z) = \sum_{j=0}^{n} \eta_j f_j(z)$, where z is the complex variable $x + iy$, $\eta_j = \alpha_j + i\beta_j$ and $\{\alpha_j\}_{j=0}^{n}$ and $\{\beta_j\}_{j=0}^{n}$ are sequences of standard normal independent random variables, and $\{f_j\}_{j=0}^{n}$ is a sequence of given analytic functions that are real-valued on the real number line. In addition, the numerical computations of the intensity functions and the empirical distributions for the special cases of random Weyl polynomials, random Taylor polynomials and random truncated Fourier cosine series are included as examples.

1. Introduction and statement of results

A random polynomial is a polynomial whose coefficients are given according to some probability distribution. As the coefficients are random variables, it is of interest to see how the zeros of the polynomial are distributed. The problem of characterizing the distribution of zeros of random polynomials has a long history. It started in 1932 with the work of Bloch and Pólya [5]. However, Kac [14,15] was the first to study, in 1943, the distribution of the real zeros of a random polynomial whose coefficients are real standard normal independent random variables and to obtain an explicit formula for the expected value of the number of its zeros in any measurable subset of the reals. Following Kac's initial investigation, the distribution of the number of real zeros of random polynomials was studied by several authors, and in particular the expected number of real zeros was considered for several different distributions besides the normal law for the coefficients. (See [4,7–10,16] and [18] and the references therein.)

In 1995 Shepp and Vanderbei [20] developed a remarkable method based on Cauchy's argument principle for extending Kac's result and derived explicit formulas for the expected number of complex zeros in any measurable subset of the complex plane, when the coefficients are real standard normal independent random

E-mail address: Andrew-Ledoan@utc.edu.

http://dx.doi.org/10.1016/j.jmaa.2016.07.015
0022-247X/

variables. They produced compelling computer plots of the intensity functions and hundreds of thousands of zeros from randomly generated polynomials, using an innovative robust zero-finding numerical algorithm, that show that, as the degrees of the random polynomials become large, the zeros tend to lie very close to the unit circle and, what is first noticeable when the real zeros are ignored, appear to be approximately uniformly distributed around the circle. In addition, the asymptotics of the intensity functions confirm the classical result due to Hammersley [11].

In 1997 Ibragimov and Zeitouni [13] employed a different method to obtain the results in [20]. Furthermore, they demonstrated the limiting distributions of the intensity functions under more general distributional assumptions.

Recently, research in random polynomials have branched off in a number of directions. For example, the zeros of many ensembles of random polynomials have been found to be asymptotically equidistributed near the unit circumference. In 2015 Pritsker and Yeager [19] derived quantitative estimates for such equidistribution in terms of the expected discrepancy of a certain zero counting measure and the expected number of zeros in various subsets of the complex plane. To be precise, they studied random polynomials whose coefficients may be dependent and need not have identical distributions.

Also in 2015 Vanderbei [23] introduced a modest generalization to the central assumptions underlying the results in [20] and showed that comparable explicit formulas for the distribution of the zeros in the complex plane can still be obtained for any value of the degree of the random polynomial. He pointed out that, for many years now, most authors establish certain properties of the zeros under very general distributional assumptions. Hence, the cost for that generality is that most results only hold asymptotically as the degree of the random polynomial tends to infinity.

Vanderbei's result can be summarized as follows: *Let*

$$S_n(z) = \sum_{j=0}^{n} \eta_j f_j(z),$$

where z *is the complex variable* $x + iy$, $\{\eta_j\}_{j=0}^n$ *is a sequence of real standard normal independent random variables defined on a probability space* $(\Omega, \mathscr{A}, \mathrm{Pr})$, *and* $\{f_j\}_{j=0}^n$ *is a sequence of given analytic functions that are real-valued on the real number line. Let* Λ *be a measurable subset of the complex plane, and denote by* $\nu_n(\Lambda)$ *the number of complex zeros in* Λ *of* S_n*. Let, further,*

$$\begin{aligned} A_0(z) &= \sum_{j=0}^{n} f_j(z)^2, & B_0(z) &= \sum_{j=0}^{n} |f_j(z)|^2, \\ A_1(z) &= \sum_{j=0}^{n} z f_j(z) f_j'(z), & B_1(z) &= \sum_{j=0}^{n} z \overline{f_j(z)} f_j'(z), \\ A_2(z) &= \sum_{j=0}^{n} z^2 f_j'(z)^2, & B_2(z) &= \sum_{j=0}^{n} |z f_j'(z)|^2, \end{aligned}$$

and define

$$D_0(z) = \sqrt{B_0(z)^2 - |A_0(z)|^2}$$

and

$$E_1(z) = \sqrt{A_0(z) A_2(z) - A_1(z)^2}.$$

For any measurable region Λ *in the complex plane and for each integer* $n > 1$,