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Relating coefficients of expansion of a function to its norm

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ABSTRACT

Relations of the coefficients of the expansion of a function by orthogonal sequence with the norm of that function are given. The results contain Hausdorff–Young and Hardy–Littlewood type inequalities. The many applications that show the use of the results and their values are crucial. Many of the applications are for expansion by orthogonal polynomials on domain D and weight w. For $w = 1$ the results are given for polytopes and other bounded convex domains. For the ball, simplex and the cube the weights dealt with are Jacobi-type. For $D = R$, R_+, and R^d $(d > 1)$ we give results using Freud (including Hermite) weights, Laguerre weights and Hermite weights respectively. Expansions by trigonometric polynomials on the torus and by spherical harmonic polynomials on the sphere are also dealt with.

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1. Introduction and setup

On domain $D \subset R^d$ and weight $w(\boldsymbol{x}) \equiv \widetilde{w}(\boldsymbol{x})^2$ satisfying $0 < w(\boldsymbol{x}) < \infty$ a.e., we set $\{\varphi_{n,\ell}(\boldsymbol{x})\}_{n=0,\ell=1}^{\infty,d_n}$ with finite d_n to be a complete orthonormal system, that is

$$\int_D \varphi_{n,\ell}(\boldsymbol{x})\varphi_{m,k}(\boldsymbol{x})w(\boldsymbol{x})d\boldsymbol{x} = \begin{cases} 1 & \text{if } n = m,\ \ell = k \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

and

$$\int_D f(\boldsymbol{x})\varphi_{n,\ell}(\boldsymbol{x})w(\boldsymbol{x})d\boldsymbol{x} = 0 \text{ for all } n \text{ and } \ell \text{ implies } f = 0. \tag{1.2}$$

In this paper f and $\varphi_{n,k}$ are real functions but most of the results carry over without much change to complex valued functions.

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The Fourier expansion of f by $\varphi_{n,\ell}$ is denoted by

$$f \sim \sum_{n=0}^{\infty} \sum_{\ell=1}^{d_n} a_{n,\ell}\varphi_{n,\ell} \text{ where } \int_D f(\boldsymbol{x})\varphi_{n,\ell}(\boldsymbol{x})w(\boldsymbol{x})d\boldsymbol{x} \equiv a_{n,\ell}. \tag{1.3}$$

We define the d_n dimensional space H_n by

$$H_n = \text{ span}\{\varphi_{n,\ell},\, 1 \le \ell \le d_n\}. \tag{1.4}$$

Examples of H_n are polynomials (algebraic, trigonometric or spherical harmonic) of degree n orthogonal with respect to D and $w(\boldsymbol{x})$ to polynomials of degree $n-1$. There are, of course, many other possibilities, some of which will be discussed in the sections dedicated to applications (see Section 8).

The $L_{p,w}(D)$ norm of f is given as usual by

$$\|f\|_{L_{p,w}(D)} = \Big\{ \int_D |f(\boldsymbol{x})|^p w(\boldsymbol{x})d\boldsymbol{x} \Big\}^{1/p}, \quad 1 \le p < \infty \tag{1.5}$$

and $\|f\|_{L_\infty(D)} = \operatorname*{ess\,sup}_{\boldsymbol{x}\in D} |f(\boldsymbol{x})|$. When $w(\boldsymbol{x}) = 1$, we write $\|f\|_{L_{p,1}(D)} = \|f\|_{L_p(D)}$. Recalling $\widetilde{w}(\boldsymbol{x})^2 = w(\boldsymbol{x})$, we have $\|f\|_{L_{2,w}(D)} = \|\widetilde{w}f\|_{L_2(D)}$.

To relate $\{a_{n,k}\}$ of (1.3) to $\|f\|_{L_{p,w}(D)}$, we will use the inequalities

$$\|g\|_{L_\infty(D)} \le C(n+1)^{\sigma/2}\|g\|_{L_{2,w}(D)}, \quad g \in \text{ span}\{H_k\}_{k=0}^n \tag{1.6}$$

and

$$\|g\|_{L_\infty(D)} \le C_1(n+1)^{\sigma_1/2}\|g\|_{L_{2,w}(D)}\,, \quad g \in H_n \tag{1.7}$$

where C and C_1 are independent of n and g.

To relate $\{a_{n,k}\}$ of (1.3) to $\|\widetilde{w}f\|_{L_p(D)}$, we use the inequalities

$$\|\widetilde{w}g\|_{L_\infty(D)} \le C(n+1)^{\widetilde{\sigma}/2}\|\widetilde{w}g\|_{L_2(D)}\,, \quad g \in \text{ span}\{H_k\}_{k=0}^n \tag{1.8}$$

and

$$\|\widetilde{w}g\|_{L_\infty(D)} \le C(n+1)^{\widetilde{\sigma}_1/2}\|\widetilde{w}g\|_{L_2(D)}\,, \quad g \in H_n \tag{1.9}$$

where C and C_1 are independent of n and g. The inequalities (1.6)–(1.9) are the basis of the proofs of the Nikolskii-type inequalities and were calculated in many cases. In Sections 6 and 7 we use such inequalities to obtain the relations between the coefficients of expansion of a function and its norm in specific cases as a consequence of our main results i.e. Theorems 2.1 and 2.2. In Section 8 we use Theorems 2.3 and 2.4 which are in fact corollaries of Theorems 2.1 and 2.2 to obtain relations with $\|\widetilde{w}f\|_{L_p(D)}$.

We call the inequalities (1.6)–(1.9) optimal if there exist points $\boldsymbol{x}_{0,n}$ for which it is achieved; for example, (1.6) is optimal if for some $\boldsymbol{x}_{0,n}$ and some sequence $\varphi_n \in \text{ span}\{H_k\}_{k=0}^n$

$$|\varphi_n(\boldsymbol{x}_{0,n})| \ge c(n+1)^{\sigma/2}\|\varphi_n\|_{L_{2,w}(D)}$$

with $c > 0$ independent of n. In many cases one point $\boldsymbol{x}_0$ satisfies the above. The numerous applications of our main result are the raison d'être of these results and a crucial part of this paper.