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Ambarzumyan-type theorems on graphs with loops and double edges

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A B S T R A C T

In this work we consider inverse spectral problems for two types of graphs with loops and/or double edges. It is shown that analogs of Ambarzumyan's theorem are true for Sturm–Liouville problems on graphs with Neumann boundary conditions at the pendant vertices and Kirchhoff's conditions at the interior vertices.

1. Introduction

From a historical point of view, the paper [1] of Ambarzumyan may be thought to be the starting point of the inverse spectral theory aiming to reconstruct the potential from the spectrum (or spectra). Ambarzumyan proved the following theorem:

If $q \in C[0,1]$, *and* $\{n^2\pi^2 : n \in \mathbb{N} \cup \{0\}\}$ *is the spectral set of the boundary value problem*

$$-y''(x) + q(x)y(x) = \lambda y(x),\ \ y'(0) = y'(1) = 0,$$

then $q(x) \equiv 0$ *in* $[0,1]$.

Later it became clear that the case investigated by Ambarzumyan was exceptional; in general, two spectra are needed to determine the potential [2,3,17]. Various generalizations of Ambarzumyan's theorem can be found in [6–9,12,14,16,15,19,23,24,26] and other works. Here we mention Ambarzumyan-type theorems on star-graphs [19,23], and Ambarzumyan-type theorems on trees [6,16].

A quantum graph is a metric graph equipped with a differential operator acting on suitable domain, consisting of functions satisfying certain boundary or matching conditions at the vertices. Differential operators

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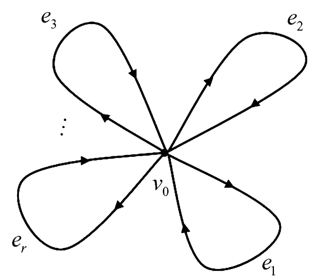

Fig. 1. Oriented graph with loops.

defined on quantum graphs are a rather new and rapidly developing area of modern mathematical physics. Such operators can be used to model the motion of quantum particles confined to certain low dimensional structures. Inverse spectral problems of Sturm–Liouville equations on graphs were investigated by Brown and Weikard [4], Carlson [5], Kuchment [13], Pivovarchik [18], Rundell and Sacks [22], and Yurko [25], etc.

In the present paper we consider inverse spectral problems for Sturm–Liouville equations on graphs with loops and/or double edges. For inverse problems on such graphs, in general, one spectrum does not uniquely determine the potential on the edges of the graph [13,18]. However, as in the case of a single interval, there is an exceptional case, in which the potential is uniquely determined by one spectrum. We consider exceptional cases in which a part of one spectrum of a boundary value problem with Neumann boundary conditions at the pendant vertices uniquely determines the set of potentials on the edges of the graphs. The proofs use the Gelfand–Levitan equation and a variational principle [10,14,17,21].

In the paper [19] a star-shaped graph consisting of three segments was considered with Neumann boundary conditions at the pendant vertices and Kirchhoff's conditions at the central vertex. The paper [23] extends the results in [19] to star graphs consisting of an arbitrary number of segments with Neumann boundary conditions and Dirichlet boundary conditions at the pendant vertices, respectively. In [6] Ambarzumyan-type theorem is generalized to Schrödinger operators on metric trees, with Neumann boundary conditions at the pendant vertices. For a Neumann Ambarzumyan problem on trees with arbitrary lengths Ambarzumyan-type theorem is also true [16].

Ambarzumyan-type problems for differential operators on graphs with loops and/or double edges have not been studied yet. It turns out that Ambarzumyan-type theorems also remain valid in the more general situation under consideration. In the subsequent sections we shall consider Ambarzumyan-type problems for differential operators on a flower-like graph with loops, and a graph with loops and double edges, respectively.

2. Ambarzumyan-type theorems on a flower-like graph with loops

2.1. Flower-like graph with loops

Consider a compact graph G in $\mathbf{R}^m$ with a vertex v_0 and the set of edges $\mathcal{E} = \{e_1, \dots, e_r\}$, where v_0 is the internal vertex, and e_j $(j = \overline{1,r})$ are all loops. Thus, the graph G has r loops e_j and one internal vertex v_0. We suppose that the length of each edge is equal to 1. Each edge $e_j \in \mathcal{E}$ is parameterized by the parameter $x \in [0,1]$; below we identify the value x of the parameter with the corresponding point on the edge. It is convenient for us to choose the following orientations: for $j = \overline{1,r}$, both ends $x = +0$ and $x = 1-0$ correspond to v_0 (see Fig. 1).

An integrable function Y on G may be represented as $Y = \{y_j\}_{j=\overline{1,r}}$, where the function $y_j(x), x \in [0,1]$, is defined on the edge e_j. Let $q = \{q_j\}_{j=\overline{1,r}}$ be a square integrable real-valued function on G. Consider the following differential equations on G:

$$-y_j''(x) + q_j(x)y_j(x) = \lambda y_j(x), \quad j = \overline{1,r}, \tag{2.1}$$