J. Math. Anal. Appl. 444 (2016) 1438–1461

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Sharp regularity and Cauchy problem of the spatially homogeneous Boltzmann equation with Debye–Yukawa potential

Léo Glangetas [b], Hao-Guang Li [a,*]

[a] School of Mathematics and Statistics, South-Central University for Nationalities, 430074 Wuhan, PR China
[b] Université de Rouen, CNRS UMR 6085, Mathématiques, 76801 Saint-Etienne du Rouvray, France

ARTICLE INFO

Article history:
Received 15 December 2015
Available online 22 July 2016
Submitted by X. Zhang

Keywords:
Boltzmann equation
Spectral decomposition
Debye–Yukawa potential

ABSTRACT

In this paper, we study the Cauchy problem for the linear spatially homogeneous Boltzmann equation with Debye–Yukawa potential. Using the spectral decomposition of the linear operator, we prove that, for an initial datum in the sense of distribution which contains the dual of the Sobolev spaces, there exists a unique solution which belongs to a more regular Sobolev space for any positive time. We also study the sharp regularity of the solution.

1. Introduction and main results

In this work, we consider the spatially homogeneous Boltzmann equation

$$\frac{\partial f}{\partial t} = Q(f, f) \tag{1.1}$$

where $f = f(t, v)$ is the density distribution function depending only on two variables $t \geq 0$ and $v \in \mathbb{R}^3$. The Boltzmann bilinear collision operator is given by

$$Q(g, f)(v) = \int_{\mathbb{R}^3}\int_{S^2} B(v - v_*, \sigma)\left(g(v'_*)f(v') - g(v_*)f(v)\right) dv_* d\sigma,$$

where for $\sigma \in \mathbb{S}^2$, the symbols v'_* and v' are abbreviations for the expressions,

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma,$$

* Corresponding author.
E-mail addresses: leo.glangetas@univ-rouen.fr (L. Glangetas), lihaoguang@mail.scuec.edu.cn (H.-G. Li).

http://dx.doi.org/10.1016/j.jmaa.2016.06.039
0022-247X/

which are obtained in such a way that collision preserves momentum and kinetic energy, namely

$$v'_* + v' = v + v_*, \quad |v'_*|^2 + |v'|^2 = |v|^2 + |v_*|^2.$$

For monatomic gas, the collision cross section $B(v - v_*, \sigma)$ is a non-negative function which depends only on $|v - v_*|$ and $\cos\theta$ which is defined through the scalar product in $\mathbb{R}^3$ by

$$\cos\theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma.$$

Without loss of generality, we may assume that $B(v - v_*, \sigma)$ is supported on the set $\cos\theta \geq 0$, i.e. where $0 \leq \theta \leq \frac{\pi}{2}$. See for example [12,26] for more explanations about the support of θ. For physical models, the collision cross section usually takes the form

$$B(v - v_*, \sigma) = \Phi(|v - v_*|) b(\cos\theta),$$

with a kinetic factor

$$\Phi(|v - v_*|) = |v - v_*|^\gamma, \ \gamma \in]-3, +\infty[.$$

The molecules are said to be Maxwellian when the parameter $\gamma = 0$.

Except for the hard sphere model, the function $b(\cos\theta)$ has a singularity at $\theta = 0$. For instance, in the important model case of the inverse-power potentials,

$$\phi(\rho) = \frac{1}{\rho^r}, \text{ with } r > 1,$$

with ρ being the distance between two interacting particles in the physical 3-dimensional space $\mathbb{R}^3$,

$$b(\cos\theta)\sin\theta \approx K\theta^{-1-\frac{2}{r}}, \text{ as } \theta \to 0^+.$$

The notation $a \approx b$ means that there exist positive constants $C_2 > C_1 > 0$, such that

$$C_1 a \leq b \leq C_2 a.$$

Notice that the Boltzmann collision operator is not well defined for the case when $r = 1$ corresponding to the Coulomb potential.

If the inter-molecule potential satisfies the Debye–Yukawa type potentials where the potential function is given by

$$\phi(\rho) = \frac{1}{\rho e^{\rho^s}}, \text{ with } s > 0,$$

then the collision cross section has a singularity in the following form

$$b(\cos\theta) \approx \theta^{-2}(\log\theta^{-1})^{\frac{2}{s}-1}, \text{ when } \theta \to 0^+, \text{ with } s > 0. \tag{1.2}$$

This explicit formula was first appeared in the Appendix in [19]. In some sense, the Debye–Yukawa type potentials is a model between the Coulomb potential corresponding to $s = 0$ and the inverse-power potential. For further details on the physics background and the derivation of the Boltzmann equation, we refer the reader to the extensive expositions [3,26].