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Extended eigenvalues for bilateral weighted shifts

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ABSTRACT

A complex scalar λ is said to be an extended eigenvalue for an operator A on a Hilbert space H if there is a non-zero operator X such that $AX = \lambda XA$, and in that case, X is said to be an extended eigenoperator. It is shown that if a bilateral weighted shift has a non-unimodular extended eigenvalue then every extended eigenoperator for A is strictly lower triangular. Also, it is shown that the set of the extended eigenvalues for an injective bilateral weighted shift is either $\mathbb{C}\setminus\mathbb{D}$ or $\mathbb{C} \setminus \{0\}$ or $\overline{\mathbb{D}} \setminus \{0\}$, or $\mathbb{T}$, and some examples are constructed in order to show that each of the four shapes does happen. Further, it is shown that the set of the extended eigenvalues for an injective bilateral weighted shift with an even sequence of weights is either $\mathbb{C}\setminus\{0\}$ or $\mathbb{T}$, and that the set of the extended eigenvalues for an invertible bilateral weighted shift is $\mathbb{T}$. Finally, a factorization result is provided for the extended eigenoperators corresponding to a unimodular extended eigenvalue of an injective bilateral weighted shift.

1. Introduction

Let A and B be two bounded operators on an infinite-dimensional, separable complex Hilbert space H. A non-zero operator X is said to *intertwine* A *and* B provided that $AX = XB$. Notice that this relation is neither symmetric nor transitive because X is not necessarily invertible.

A complex scalar λ is said to be an *extended eigenvalue* for an operator A provided that there is a non-zero operator X that intertwines A and λA, that is, $AX = \lambda XA$. Such an operator X is said to be a corresponding *extended eigenoperator* associated with λ.

Scott Brown [5] and simultaneously Kim, Moore and Pearcy [6] discovered the following generalization of Victor Lomonosov's famous invariant subspace theorem [12], namely, that if a non-scalar operator A has a non-zero, compact extended eigenoperator then A has a non-trivial, closed hyperinvariant subspace.

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The notion of an extended eigenvalue has generated some literature ever since then, both in the search of invariant subspaces [2,8–10] and in the computation of the extended eigenvalues for some special classes of operators [1,3,4,7,11,13,15].

Let (e_n) be an orthonormal basis of H and consider the weighted shift operator A defined on the elements of this basis by the expression

$$Ae_n = \alpha_n e_{n+1}, \tag{1.1}$$

where (α_n) is a bounded sequence of non-zero scalars. If n runs through the natural numbers then A is called a *unilateral weighted shift* and if n runs through the integers then A is called a *bilateral weighted shift*. The classical survey by Allen Shields [14] is a nice source of information about weighted shift operators.

Srdjan Petrovic [13] showed that if A is an injective unilateral shift then every $\lambda \in \mathbb{C}\backslash\mathbb{D}$ is an extended eigenvalue for A. He also showed that if A has some extended eigenvalue $\lambda_0 \in \mathbb{D}\backslash\{0\}$ then *every* $\lambda \in \mathbb{D}\backslash\{0\}$ is an extended eigenvalue for A, so that the set of the extended eigenvalues for A is either $\mathbb{C}\backslash\mathbb{D}$ or $\mathbb{C}\backslash\{0\}$. He also constructed examples in order to show that each of the two shapes does happen.

Let us denote by $\mathrm{Ext}(A)$ the set of the extended eigenvalues for a Hilbert space operator A. The first author et al. [7] considered an injective bilateral weighted shift A and they made the observation that $\mathrm{Ext}(A)$ has circular symmetry, that is, if $\lambda \in \mathrm{Ext}(A)$ then $e^{i\theta}\lambda \in \mathrm{Ext}(A)$ for all $\theta \in \mathbb{R}$, and as a consequence, $\mathbb{T} \subseteq \mathrm{Ext}(A)$. They also showed that $\mathrm{Ext}(A) = \mathbb{T}$ when the point spectrum of A has non-empty interior, and moreover, every extended eigenoperator X corresponding to an extended eigenvalue $\lambda \in \mathbb{T}$ factors as $X = D_\lambda B$ for some $B \in \{A\}'$, where D_λ is a particular extended eigenoperator corresponding to λ, namely, the diagonal operator defined by the expression

$$D_\lambda e_n = \lambda^{-n} e_n. \tag{1.2}$$

The aim of this paper is to describe the extended eigenvalues and to characterize the corresponding extended eigenoperators for an injective bilateral weighted shift. The paper is organized as follows.

In section 2 we provide a necessary and sufficient condition for the existence of a non-zero intertwining operator for two injective bilateral weighted shifts.

In section 3 we consider an injective bilateral weighted shift A and we show that if there is $\lambda \in \mathrm{Ext}(A)$ with $|\lambda| \neq 1$ then every extended eigenoperator for A corresponding to λ is strictly lower triangular, and moreover, $\mathrm{Ext}(A)$ must be either $\mathbb{C} \setminus \mathbb{D}$ or $\mathbb{C} \setminus \{0\}$ or $\overline{\mathbb{D}} \setminus \{0\}$, or $\mathbb{T}$.

In section 4 we construct some examples of injective bilateral weighted shifts A in order to show that $\mathrm{Ext}(A)$ can be either $\mathbb{C} \setminus \mathbb{D}$ or $\mathbb{C} \setminus \{0\}$ or $\overline{\mathbb{D}} \setminus \{0\}$, or $\mathbb{T}$. We also show that if A is an invertible operator then $\mathrm{Ext}(A) \subseteq \mathbb{T}$, and as a consequence, if A is an invertible bilateral weighted shift then $\mathrm{Ext}(A) = \mathbb{T}$. Further, we show that if A is an injective bilateral weighted shift with an even sequence of weights then either $\mathrm{Ext}(A) = \mathbb{T}$ or $\mathrm{Ext}(A) = \mathbb{C}\backslash\{0\}$.

In section 5 we extend the factorization result of the first author et al. [7] to the general case when no restrictions are made on the point spectrum of A. We also show that if a bilateral weighted shift A has a non-zero, compact extended eigenoperator corresponding to a unimodular extended eigenvalue then A commutes with a non-zero compact operator.

2. Intertwining relations for bilateral weighted shifts

Allen Shields [14, Proposition 5] proved that an operator X intertwines two bilateral weighted shifts A and B with sequences of weights $(\alpha_n)_{n\in\mathbb{Z}}$ and $(\beta_n)_{n\in\mathbb{Z}}$ if and only if

$$\beta_j x_{i+1,j+1} = \alpha_i x_{i,j}, \tag{2.1}$$