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On the lack of semiconcavity of the subRiemannian distance in a class of Carnot groups

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ABSTRACT

We show by explicit estimates that the SubRiemannian distance in a Carnot group of step two is locally semiconcave away from the diagonal if and only if the group does not contain abnormal minimizing curves. Moreover, we prove that local semiconcavity fails to hold in the step-3 Engel group, even in the weaker "horizontal" sense.

1. Introduction

It is well known that subRiemannian spheres are rather irregular objects. Already in the simplest example—the Heisenberg group—the subRiemannian distance from the origin is only Lipschitz-continuous at points of the center of the group. Furthermore, it can be shown that the only subRiemannian manifolds where (small) spheres are smooth are the Riemannian ones (see [3]).

The irregularity of the distance function is mainly governed by the presence of *abnormal geodesics* (see Section 2). Indeed, the function $d(x_0, \cdot)$ can not be smooth at any point x connected to x_0 by an abnormal length-minimizer (see [3]). Furthermore, it has been shown in several papers by Agrachev, Bonnard, Chyba and Kupka [4], Trélat [27] and Agrachev [2] that, under the *corank 1* assumption, where in particular all abnormal extremals are *strictly abnormal*, at a point x along an abnormal length-minimizing curve γ leaving from x_0, the subRiemannian sphere centered at x_0 is tangent to γ in a suitable sense and ultimately the distance from x_0 can not be expected to be even Lipschitz at x.

On the other side, it is known that abnormal minimizers do not appear at all for a subclass of two-step Carnot groups (Métivier groups) and, by a result of Chitour, Jean and Trélat [12], in the very large class furnished by *generic* subRiemannian structures of rank at least three.

In the papers [10,13], Cannarsa and Rifford, and Figalli and Rifford showed that in a bracket generating subRiemannian manifold where all length-minimizing paths are strictly normal, the subRiemannian distance

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from a fixed base point $x_0 \in M$ is locally semiconcave in $M \setminus \{x_0\}$. Since local semiconcavity implies local Lipschitz-continuity, this result can not be extended to the situation where corank 1 abnormal minimizers appear.

However, there are subRiemannian manifolds and more specifically Carnot groups which do not belong to the class in [10,13], because they contain abnormal minimizing paths, but do not enjoy the corank 1 assumption of [27] and [2], because abnormal minimizing paths are normal too (we say that they are *normal-abnormal*). This class includes all non-Métivier two-step Carnot groups and some step-three Carnot groups.

In this paper we show some negative results on the local semiconcavity of subRiemannian distances in the setting of non-Métivier two-step groups and in the step-three Engel group. We also discuss a weaker property, namely the *horizontal semiconcavity* and we show that, in all two-step free groups, such property holds "pointwise" at all abnormal points, where the usual Euclidean notion fails to hold. We plan to come back to a detailed study of local horizontal semiconcavity for the distance in two-step Carnot groups in a subsequent work. On the other side, it turns out that in the three-step Engel group the horizontal semiconcavity fails to hold.

Besides its relevant role in the optimal transport problems studied in [13], local semiconcavity of the subRiemannian distance plays a role in the construction of suitable "barrier functions" in potential theory which are a fundamental tool in the study of second order nondivergence subelliptic PDEs with measurable coefficients (see [14,26,21]).

To state our result, we also introduce briefly some notation for two-step Carnot groups. Let (x, t) be coordinates in $\mathbb{R}^m \times \mathbb{R}^\ell$. Fix a family $A^1, \dots, A^\ell \in \mathbb{R}^{m\times m}$ of skew-symmetric matrices and define the composition law

$$(x,t)\cdot(\xi,\tau) = \Big(x+\xi, t+\tau+\frac{1}{2}\langle x, A\xi\rangle\Big) \tag{1.1}$$

where $\langle x, A\xi\rangle := (\langle x, A^1\xi\rangle), \dots, \langle x, A^\ell\xi\rangle) \in \mathbb{R}^\ell$ and $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^m$. We always assume the Hörmander condition $\operatorname{span}\{(A^1_{jk}, \dots, A^\ell_{jk}) : 1 \le j < k \le m\} = \mathbb{R}^\ell$ and we denote by d be the subRiemannian distance defined by the family of left-invariant vector fields $X_j = \partial_{x_j} + \frac{1}{2}\sum_{k=1}^m \sum_{\alpha=1}^\ell A^\alpha_{kj} x_k \partial_{t_\alpha}$, for $j = 1, \dots, m$. See Section 2.

Here is our statement on two-step Carnot groups, where we always denote by d the subRiemannian distance from the origin.

Theorem 1.1. *Let $(\mathbb{G}, \cdot) = (\mathbb{R}^n, \cdot) = (\mathbb{R}^m_x \times \mathbb{R}^\ell_t, \cdot)$ be the two-step Carnot group equipped with the law (1.1). Then, at any $(x, 0) = \gamma(1)$, final point of an abnormal minimizer γ leaving from the origin, there are $C > 0$ and $\tau \in \mathbb{R}^\ell$ such that we have*

$$d(x, \beta\tau) - d(x, 0) \ge C|\beta| \quad \text{for all } \beta \in [-1, 1]. \tag{1.2}$$

Moreover, if $(\mathbb{G}, \cdot) = (\mathbb{R}^n, \cdot)$ is free, then for any $(x, t) = \gamma(1)$, final point of an abnormal minimizer γ leaving from the origin, there are $C > 0$ and $(0, \tau) \in \mathbb{G}$ such that

$$d(x, t+\beta\tau) - d(x, t) \ge C|\beta| \quad \text{for all } \beta \in [-1, 1]. \tag{1.3}$$

Remark that in two-step Carnot groups abnormal minimizers are always normal (see [6, Section 20.5] or [24, Theorem 2.22]). Both estimates of this theorem ensure that the distance is not semiconcave (see the definition in (2.5)).

It is known that for step-two Carnot groups, $x \mapsto d(0, x)$ is Lipschitz for x belonging to compact sets which do not intersect the origin. Then, failure of semiconcavity can be visualized as a presence of an