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An extended Hamiltonian algorithm for the general linear matrix equation

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ABSTRACT

A second-order learning algorithm based on differential geometry is used to numerically solve the linear matrix equation $Q = x + \sum_{i=1}^{m} A_i^T x A_i - \sum_{i=1}^{n} B_i^T x B_i$. An extended Hamiltonian algorithm is proposed based on the manifold of symmetric positive definite matrices. The algorithm is compared with traditional coupled fixed-point algorithm. Numerical experiments illustrate that the convergence speed of the provided algorithm is faster than that of the coupled fixed-point algorithm.

1. Introduction

The linear matrix equation is presented as follows:

$$Q = x + \sum_{i=1}^{m} A_i^T x A_i - \sum_{i=1}^{m} B_i^T x B_i, \tag{1}$$

where Q is an $n \times n$ positive definite matrix and A_i, B_i are $n \times n$ arbitrary matrices. A_i^T and B_i^T denote the conjugate transpose of the matrices A_i and B_i, respectively. Equation (1) is the general case of the generalized Lyapunov equation as follows:

$$MYS^* + SYM^* + \sum_{k=1}^{t} N_k Y N_k^* + CC^* = 0. \tag{2}$$

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The positive definite solution of Equation (2) is the controllability Gramian of the bilinear control system (see [4,5,22] for more details)

$$M\dot{x}(t) = Sx(t) + \sum_{k=1}^{t} N_k x(t) u_k(t) + Cu(t). \tag{3}$$

Special cases of Equation (1) have been previously investigated. Several numerical methods, such as the Bartels–Stewart, Schur and QR decomposition methods, and Hessenberg–Schur, have been proposed in [3] and [12] to solve the following well-known Lyapunov equation $Q = x + A^T x A$. Some sufficient conditions for the existence of a unique symmetric positive definite solution of equations $Q = x + \sum_{i=1}^{m} A_i^T x A_i$ and $Q = x - \sum_{i=1}^{m} B_i^T x B_i$ are given in [17,18], according to the Kronecker product and the fixed point theorem in partially ordered sets. Duan used matrix differentiation to obtain a precise perturbation bound for the positive definite solution for Equation (1) [5]. Berzig provided a sufficient condition for the existence of a unique positive definite solution, and an iterative algorithm using the Bhaskar–Lakshmikantham coupled fixed-point theorem (CFPA) was proposed [2].

The solution of Equation (1) is a symmetric positive definite matrix. Moreover, the set of all the symmetric positive definite matrices can be considered to be manifold. Thus, evaluating the solution problem is facilitated using the geometric structures on this manifold [6–8]. Fiori developed the extended Hamiltonian algorithm (EHA), which is a second-order learning algorithm on manifold [11]. The inclusion of a momentum term remarkably enhances the rate of convergence [16,19]. Thus, the EHA is proposed to numerically solve Equation (1), which is based on the geometric structures of the Riemannian manifold.

Section 2 describes some fundamental concepts of the manifold discussed in the study. Section 3 introduces the EHA for Equation (1) on the manifold. Section 4 compares and demonstrates the behavior of the EHA with the CFPA using simulations.

2. Preliminaries

$\mathrm{M}(n)$ is a set of $n \times n$ real matrices and $\mathrm{GL}(n)$ is its subset, which contains only non-singular matrices. $\mathrm{GL}(n)$ is a Lie group, that is, a group which is also a differentiable manifold and on which the operations of group multiplication and the inverse are smooth. The tangent space at the identity of $\mathrm{GL}(n)$ is called the corresponding Lie algebra. This space comprises all linear transformations, namely $\mathrm{M}(n)$. Additionally, $x > 0$ implies that x is a symmetric positive definite matrix. For a different notation $x - y > 0$, $x > y$. The manifold consisting of all the symmetric positive definite matrices is defined as follows: $\mathbb{S}^+(n) = \{x \in \mathbb{R}^{n\times n} | x^T = x, x > 0\}$. The tangent space at point $x \in \mathbb{S}^+(n)$ is given by $T_x\mathbb{S}^+(n) = \{V \in \mathbb{R}^{n\times n} | v^T = v\}$. The manifold $\mathbb{S}^+(n)$ is not a Lie subgroup of $\mathrm{GL}(n)$, but it is a submanifold of $\mathrm{GL}(n)$. Different metrics on the manifold $\mathbb{S}^+(n)$ can be defined, such that different geometric structures are established on this manifold. These geometric structures, which are used in the following sections, are briefly introduced in this section. Their detailed description can be found in [1,9,14,15,20].

2.1. Tangent space on manifold $\mathbb{S}^+(n)$

The exponential map of a matrix $v \in \mathrm{M}(n)$ is usually given, by the following convergent series:

$$\exp(v) = \sum_{m=0}^{\infty} \frac{v^m}{m!}. \tag{4}$$

The inverse map, that is, the logarithmic map is defined as follows: