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On the one dimensional spectral Heat content for stable processes

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ABSTRACT

This paper provides the second term in the small time asymptotic expansion of the spectral heat content of a rotationally invariant α-stable process, $0 < \alpha \leq 2$, for the bounded interval (a, b). The small time behavior of the spectral heat content turns out to be linked to the distribution of the supremum and infimum processes.

1. Introduction

Let $0 < \alpha \leq 2$ and consider $\mathbf{X} = \{X_s\}_{s \geq 0}$ a rotationally invariant α-stable process in $\mathbb{R}^d$ where $d \geq 1$ and whose transition densities, denoted along the paper by $p_t^{(\alpha)}(x, y)$, are known to satisfy the following properties.

(i) $p_t^{(\alpha)}(x, y)$ is radial. Namely, $p_t^{(\alpha)}(x, y) = p_t^{(\alpha)}(|x - y|)$.

(ii) Scaling property: $p_t^{(\alpha)}(|x - y|) = t^{-\frac{d}{\alpha}} p_1^{(\alpha)}(t^{-\frac{1}{\alpha}} |x - y|)$.

(iii) $p_t^{(\alpha)}(x, y)$ satisfies the following two sided estimates for all $0 < \alpha < 2$. There exists a constant $c_{\alpha,d} > 0$ such that

$$c_{\alpha,d}^{-1} \min\left\{t^{-d/\alpha}, \frac{t}{|x - y|^{d+\alpha}}\right\} \leq p_t^{(\alpha)}(x - y) \leq c_{\alpha,d} \min\left\{t^{-d/\alpha}, \frac{t}{|x - y|^{d+\alpha}}\right\}, \tag{1.1}$$

for all $x, y \in \mathbb{R}^d$ and $t > 0$. See [6] for further details.

(iv) According to [3, Theorem 2.1], for all $0 < \alpha < 2$, we have

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$$\lim_{t\downarrow 0} \frac{p_t^{(\alpha)}(|x-y|)}{t} = \frac{A_{\alpha,d}}{|x-y|^{d+\alpha}}, \tag{1.2}$$

for all $x \neq y$, where

$$A_{\alpha,d} = \alpha\, 2^{\alpha-1}\, \pi^{-1-\frac{d}{2}}\, \sin\left(\frac{\pi\alpha}{2}\right) \Gamma\left(\frac{d+\alpha}{2}\right) \Gamma\left(\frac{\alpha}{2}\right). \tag{1.3}$$

Before continuing, we introduce the following standard notation. $\mathbb{E}^x$ and $\mathbb{P}^x$ will denote the expectation and probability of any process started at x, respectively. Also for simplicity, we will connote $\mathbb{P} = \mathbb{P}^0$, $\mathbb{E} = \mathbb{E}^0$ and write $Z \overset{\mathcal{D}}{=} Y$ for two random variables Z, Y with values in $\mathbb{R}^d$ to mean that they are equal in distribution or have the same law.

Let us at this point establish the following convention which is needed to provide some references and motivation. When $d = 1$, Ω will stand for an interval (a, b) with finite length $b-a$ denoted by $|\Omega|$. For $d > 1$, Ω will be a bounded domain with smooth boundary $\partial\Omega$ and volume denoted by $|\Omega|$. In addition, we set

$$|\partial\Omega| = \begin{cases} \#\{x \in \mathbb{R} : x \in \partial\Omega\}, & \text{if } d = 1, \\ \text{surface area of } \Omega, & \text{if } \ d \geq 2. \end{cases} \tag{1.4}$$

Given $\Omega \subseteq \mathbb{R}^d$ as above, we consider for $\mathbf{X} = \{X_s\}_{s\geq 0}$ the first exit time upon Ω. That is,

$$\tau_\Omega^{(\alpha)} = \inf\{s \geq 0 : X_s \in \Omega^c\}.$$

The purpose of the paper is to investigate the small time behavior of the following function, which is called *the spectral heat content over* Ω and it is given by

$$Q_\Omega^{(\alpha)}(t) = \int_\Omega dx\, \mathbb{P}^x\left(\tau_\Omega^{(\alpha)} > t\right),\ t > 0,$$

when $\Omega = (a, b)$ with $|\Omega| = b-a < \infty$. Of course, $Q_\Omega^{(\alpha)}(t)$ makes sense even in the higher dimensional setting when Ω is taken according to our convention about Ω.

It is worth noting that the *spectral heat content* of Ω takes an alternative form. In fact,

$$Q_\Omega^{(\alpha)}(t) = \int_\Omega dx \int_\Omega dy\ p_t^{\Omega,\alpha}(x, y), \tag{1.5}$$

where $p_t^{\Omega,\alpha}(x, y)$ is the transition density for the stable process killed upon exiting Ω. An explicit expression is given by

$$p_t^{\Omega,\alpha}(x, y) = p_t^{(\alpha)}(x, y)\ \mathbb{P}\left(\tau_\Omega^{(\alpha)} > t \mid X_0 = x,\ X_t = y\right). \tag{1.6}$$

The name *spectral heat content* given to $Q_\Omega^{(\alpha)}(t)$ comes from the fact that $p_t^{\Omega,\alpha}(x, y)$ can be written in terms of the eigenvalues and eigenfunctions of the underlying domain Ω. That is, it is known (see [8]) that there exists an orthonormal basis of eigenfunctions $\{\phi_n\}_{n\in\mathbb{N}}$ for $L^2(\Omega)$ with corresponding eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ satisfying $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ and $\lambda_n \to \infty$ as $n \to \infty$ such that

$$p_t^{\Omega,\alpha}(x, y) = \sum_{n=1}^{\infty} e^{-t\lambda_n}\, \phi_n(x)\, \phi_n(y).$$