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Heat content estimates over sets of finite perimeter

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ABSTRACT

This paper studies by means of standard analytic tools the small time behavior of the heat content over a bounded Lebesgue measurable set of finite perimeter by working with the set covariance function and by imposing conditions on the heat kernels. Applications concerning the heat kernels of rotational invariant α-stable processes are given.

1. Introduction

Let I be a set of indices and $d \geq 2$ an integer. Consider a set of non-negative functions

$$\left\{p_t^{(\alpha)}(\cdot) : \mathbb{R}^d \to [0,\infty], \alpha \in I, t \geq 0\right\},$$

where each $p_t^{(\alpha)}(\cdot)$ will be called *heat kernel*. We shall assume that these heat kernels satisfy the following properties.

(i) For each $t > 0$, $p_t^{(\alpha)}(x)$ is radial. That is, $p_t^{(\alpha)}(x) = p_t^{(\alpha)}(|x|) \geq 0$, $x \in \mathbb{R}^d$. Furthermore, we assume $p_t^{(\alpha)}(\cdot) \in L^1(\mathbb{R}^d)$.

(ii) Scaling property: for each integer $d \geq 2$ and $\alpha \in I$, there exist $\beta = \beta(d,\alpha) \in \mathbb{R}$ and $\gamma = \gamma(d,\alpha) > 0$ such that

$$p_t^{(\alpha)}(x) = t^{\beta} p_1^{(\alpha)}(t^{-\gamma}x). \tag{1.1}$$

As a consequence of the aforementioned properties, we obtain

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$$||p_t^{(\alpha)}||_{L^1(\mathbb{R}^d)} = t^{\beta+d\gamma}\, ||p_1^{(\alpha)}||_{L^1(\mathbb{R}^d)}, \tag{1.2}$$
$$p_t^{(\alpha)}(x) = p_t^{(\alpha)}(|x|\, e_d),$$

where e_d stands for the vector $(0, 0, \ldots, 0, 1) \in \mathbb{R}^d$.

Before continuing, we provide some useful notations. Throughout the paper, $\mathcal{L}(\mathbb{R}^d)$ will denote the set of all the Lebesgue measurable subsets of $\mathbb{R}^d$. For a bounded set $\Omega \in \mathcal{L}(\mathbb{R}^d)$ with non-empty boundary $\partial\Omega$, we set

$$|\Omega| = \text{volume of } \Omega,$$
$$\mathcal{H}^{d-1}(\partial\Omega) = (d-1)\text{-Hausdorff measure of the boundary of } \Omega.$$

Henceforth, $B_r(x)$ will stand for the ball centered at $x \in \mathbb{R}^d$ with radius r and for simplicity B will represent the unit ball centered at zero. Also S^{d-1} will denote the boundary of the unit ball B. Moreover, the volume and surface area of the unit ball in $\mathbb{R}^d$ will be denoted by w_d and A_d, respectively. That is,

$$w_d = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(1 + \frac{d}{2}\right)}, \tag{1.3}$$
$$A_d = d w_d.$$

In addition, if $g : \Omega \subseteq \mathbb{R}^d \to \mathbb{R}$ is a Lipschitz function, we denote

$$Lip(g) = \sup\left\{\frac{|g(y) - g(x)|}{|y - x|} : x, y \in \Omega, x \neq y\right\}.$$

Let $\Omega \in \mathcal{L}(\mathbb{R}^d)$ be a bounded set. The purpose of the paper is to investigate the behavior as $t \to 0+$ of the following function

$$\mathbb{H}_\Omega^{(\alpha)}(t) = \int_\Omega dx \int_\Omega dy\, p_t^{(\alpha)}(x - y), \tag{1.4}$$

which will be called *the heat content* of Ω in $\mathbb{R}^d$ by imposing conditions over the heat kernel $p_t^{(\alpha)}(\cdot)$ and the underlying set Ω. We remark that $\mathbb{H}_\Omega^{(\alpha)}(t)$ is finite for all $t > 0$ due to the assumption $p_t^{(\alpha)}(\cdot) \in L^1(\mathbb{R}^d)$ and the inequality

$$0 \leq \mathbb{H}_\Omega^{(\alpha)}(t) \leq \int_\Omega dx \int_{\mathbb{R}^d} dy\, p_t^{(\alpha)}(x - y) = |\Omega|\; ||p_t^{(\alpha)}||_{L^1(\mathbb{R}^d)}.$$

The function $\mathbb{H}_\Omega^{(\alpha)}(t)$ turns out to provide information about the geometry of the set Ω as long as regularity conditions over Ω are assumed. For instance, in [20], Theorem 2.4 is proved by taking $I = \{2\}$ and considering the Gaussian kernel

$$p_t^{(2)}(x) = (4\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{4t}\right) \tag{1.5}$$

that

$$\lim_{t \to 0+} \frac{|\Omega| - \mathbb{H}_\Omega^{(2)}(t)}{\sqrt{t}} = \frac{1}{\sqrt{\pi}} \mathcal{H}^{d-1}(\partial\Omega),$$