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Bifurcation of stable equilibria under nonlinear flux boundary condition with null average weight

Gustavo Ferron Madeira *, Arnaldo Simal do Nascimento

Departamento de Matemática, Universidade Federal de São Carlos, 13.565-905 São Carlos, SP, Brazil

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ABSTRACT

The bifurcation and stability structures of equilibria of a parabolic problem motivated by a population genetics model are completely described. In a previous work the authors considered weights having nonzero average and drew the bifurcation and stability diagrams of equilibria. Herein that study is completed by considering weights with null average over the boundary of the domain and other approach is employed to overcome a degeneration which occurs in such case. For each value of the parameter we prove existence of an unique nonconstant equilibrium solution. It belongs to a globally parametrized bifurcation curve constructed using a Lyapunov–Schmidt type reduction combined with the Morse lemma. That solution is showed to be a global minimizer of the corresponding energy functional which attracts all nontrivial semi-orbits. The behavior of its trace, when the parameter is large, is established and complete diagrams containing all bifurcation and stability information are also provided.

1. Introduction

The aim of this work is to study the bifurcation and stability structures of equilibria of an evolution problem whose motivation comes mainly from population genetics. More precisely, we consider the following parabolic problem under a nonlinear Neumann boundary condition

$$\begin{cases} \partial_t u = \Delta u & \text{in} \quad \Omega \times \mathbb{R}^+ \\ \dfrac{\partial u}{\partial \nu} = \lambda a(x) f(u) & \text{on} \quad \partial\Omega \times \mathbb{R}^+ \\ u(\cdot, 0) \in \mathfrak{X}. \end{cases} \tag{1}$$

In (1), $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a smooth bounded domain with outward normal vector ν and $\lambda > 0$ is a parameter. The boundary condition means that the flux across the boundary in the outward normal direction

* Corresponding author.
E-mail addresses: gfmadeira@dm.ufscar.br (G.F. Madeira), arnaldon@dm.ufscar.br (A.S. do Nascimento).

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is proportional to the product of a prescribed function of the frequency or density u with a weight function $a(\cdot)$ of indefinite sign.

The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is smooth, say of class C^4, and satisfies

$$(\boldsymbol{H_1}) \quad \begin{cases} f > 0 \quad \text{in} \quad (0,1), & f(0) = 0 = f(1), \\ f'(0) > 0, \quad f'(1) < 0, & f''(u) < 0 \quad \text{in} \quad (0,1). \end{cases}$$

Examples of functions satisfying $(\boldsymbol{H_1})$ are the logistic functions $f(u) = u(1-u)$ or $f(u) = u(1-u)(hu + (1-h)(1-u))$, where h is a constant in the range $1/3 \leq h \leq 2/3$, which appear e.g., in population genetics problems (see [5,8,9,12,13] and references therein).

The indefinite weight $a : \partial\Omega \longrightarrow \mathbb{R}$ is of class $C^{1,\theta}(\partial\Omega)$, $0 < \theta < 1$, such that

$$(\boldsymbol{H_2}) \quad \int_{\partial\Omega} a(x)\, d\mathcal{H}^{n-1} = 0.$$

Moreover, we have

$(\boldsymbol{H_3})$ $\quad \mathfrak{X} \doteq \left\{ u \in H^1(\Omega) \ : \ 0 \leq u(x) \leq 1 \text{ a.e. } x \in \Omega \right\}$ is the phase space for (1)

keeping analogy with several problems occurring in population genetics where solutions satisfying $0 \leq u \leq 1$ are of interest.

Problem (1) was studied in [19] and was motivated by the selection–migration model for alleles in a given region of space introduced by Fisher [8] and generalized by Fleming [9] and Henry [12]. That model describes the changes of gene frequency for a population confined in Ω considering natural selection effects in Ω only and no flux through $\partial\Omega$. Further, it gives rise to a parabolic equation supplied with a homogeneous Neumann boundary condition appearing in [9,12], which is studied under various aspects by several authors, for instance, [25,13–15,26,21,16].

As a consequence of the maximum principle one can prove that (1) generates a dynamical system in $\mathfrak{X}$, which is a gradient system (see [17]). Such information sets the equilibrium solutions or steady-state solutions in evidence from the large time dynamic viewpoint. The equilibrium solutions to (1) are the solutions to

$$\begin{cases} \Delta u = 0 \quad \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda a(x) f(u) \quad \text{on} \quad \partial\Omega \\ u \in \mathfrak{X}. \end{cases} \tag{2}$$

There is a large number of works dealing with elliptic and parabolic problems under nonlinear boundary conditions. We mention some of them whose results or approaches are closer to this paper. For instance, [1,6,18] on existence of nonconstant stable equilibrium solutions to parabolic problems under nonlinear boundary conditions. For results on bifurcation and/or stability of equilibria for such type of problems, see e.g. [2,3,10,19,23,24,28,29,32,30,31]. The reader can also see the references cited in those papers.

In [19] the authors considered (1) under hypotheses $(\boldsymbol{H_1})$ and $(\boldsymbol{H_3})$ but, instead of $(\boldsymbol{H_2})$, required the weight $a(\cdot)$ to be of indefinite sign with nonzero average over $\partial\Omega$. For such situation it was established in the mentioned paper the bifurcation and stability structures of equilibria to (1). Indeed, it was proved the existence of $\lambda_0 > 0$ such that there is no nonconstant solution to (2) for $0 < \lambda < \lambda_0$, while a global branch consisting of exponentially stable equilibria to (1) bifurcates from $(\lambda_0, 0)$ or $(\lambda_0, 1)$ according to the sign of $\int_{\partial\Omega} a\, d\mathcal{H}^{n-1}$. Furthermore such curves are global smooth curves of the parameter λ. Those conclusions are expressed in the schematic diagrams in Fig. 1 and Fig. 2 below.

Now if $a(\cdot)$ has null average over $\partial\Omega$ the structure of the solution set of (2) described in Fig. 1 and Fig. 2 changes, as will be seen, and a different approach is necessary. Actually, investigating bifurcation and stability of equilibria to (1) under $(\boldsymbol{H_1})$–$(\boldsymbol{H_3})$, besides its behavior for large λ, is our aim in this paper.