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Measure-valued branching processes associated with Neumann nonlinear semiflows

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A B S T R A C T

We construct a measure-valued branching Markov process associated with a nonlinear boundary value problem, where the boundary condition has a nonlinear pseudo monotone branching mechanism term $-\beta$, which includes the case $\beta(u) = -u^m$, with $0 < m < 1$. The process is then used in the probabilistic representation of the solution of the parabolic problem associated with a nonlinear Neumann boundary value problem. In this way the classical association of the superprocesses to the Dirichlet boundary value problems also holds for the nonlinear Neumann boundary value problems. It turns out that the obtained branching process behaves on the measures carried by the given open set like the linear continuous semiflow, induced by the reflected Brownian motion, while the branching occurs on the measures having non-zero traces on the boundary of the open set, with the behavior of the $(-\beta)$-superprocess, having as spatial motion the process on the boundary associated to the reflected Brownian motion.

1. Introduction

Let $\mathcal{O}$ be a bounded, open subset of $\mathbb{R}^d$, $d \geqslant 1$, with smooth boundary Γ (for instance, of class C^2). Consider the nonlinear parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \frac{1}{2}\Delta u + \alpha u = 0 & \text{in } (0,\infty)\times\mathcal{O}, \\ \dfrac{\partial u}{\partial \nu} + \beta(u) = g & \text{on } \Gamma, \\ u(0,\cdot) = f & \text{in } \mathcal{O}, \end{cases} \tag{1.1}$$

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where $\frac{\partial}{\partial\nu}$ is the outward normal derivative to the boundary Γ of $\mathcal{O}$, g is a positive continuously differentiable function on Γ, $f \in C(\overline{\mathcal{O}})$, $\alpha \in \mathbb{R}_+^*$, and $\beta : \mathbb{R} \longrightarrow \mathbb{R}_-$ is the following continuous mapping

$$\beta(u) = \begin{cases} \int_0^\infty (e^{-su} - 1)\eta(ds) - bu, & \text{if } u \geqslant 0, \\ 0, & \text{if } u < 0, \end{cases} \tag{1.2}$$

with η a positive measure on $\mathbb{R}_+$ such that $\int_{\mathbb{R}_+} s \wedge 1\eta(ds) < \infty$ and $b \in \mathbb{R}_+$.

We assume that

$$\int_{\mathbb{R}_+} s \wedge 1\eta(ds) + b \leqslant \gamma := \inf_{v \in H^1(\mathcal{O})} \frac{\|\nabla v\|^2_{L^2(\mathcal{O})} + \alpha\|v\|^2_{L^2(\mathcal{O})}}{\|v\|_{L^2(\Gamma)}}. \tag{1.3}$$

Note that the last inequality is equivalent with the property of the function $u \longmapsto \beta(u) + \gamma u$ to be nondecreasing.

The function $-\beta$ is called *branching mechanism* and an example is $\beta(u) = au^m$, with $0 < m < 1$ for a convenient number $a < 0$, since

$$u^m = \frac{m}{\Gamma(1-m)} \int_0^\infty \frac{1 - e^{-su}}{s^{m+1}} ds.$$

If $\beta = \alpha = 0$ and $g = 0$ then the solution of the linear problem (1.1) is given by the transition function of the reflected Brownian motion $B = (B_t)_{t \geqslant 0}$ on $\mathcal{O}$: $u(t,\cdot) = \mathbb{E}^{\cdot}(f(B_t))$, $t \geqslant 0$, where f is a bounded, real-valued Borel measurable function on $\mathcal{O}$.

The first aim of this paper is to show that the solution of (1.1) admits a probabilistic interpretation if β is given by (1.2), which is similar with what happens in the linear case ($\beta = 0$). More precisely, there exists a branching Markov process $X = (X_t)_{t \geqslant 0}$ with state space the set $M(\overline{\mathcal{O}})$ of all positive finite measures on $\overline{\mathcal{O}}$, such that the solution of (1.1) is

$$u(t,x) = -\ln \mathbb{E}^{\delta_x}(e_f(X_t)),\ t \geqslant 0,\ x \in \overline{\mathcal{O}}, \tag{1.4}$$

where for a Borel, positive, real-valued function f on $\overline{\mathcal{O}}$ we considered the *exponential mapping* $e_f : M(\overline{\mathcal{O}}) \longrightarrow [0,1]$, defined as

$$e_f(\mu) := e^{-\int f d\mu} \quad \text{for all } \mu \in M(\overline{\mathcal{O}}).$$

The first step of our approach is to prove the existence of the solution of (1.1). We consider the maximal monotone operator $\mathcal{A}$ associated to (1.1) and we show that it is the infinitesimal generator of a nonlinear semigroup of contractions $(V_t)_{t \geqslant 0}$ on $L^2(\mathcal{O})$, such that $u(t,\cdot) := V_t f$ is a solution of (1.1) for each f from the domain of $\mathcal{A}$. If $1 \leqslant d \leqslant 3$ then $(V_t)_{t \geqslant 0}$ induces a C_0-semigroup of (nonlinear) contractions on $C(\overline{\mathcal{O}})$.

The second step is to prove that the map $f \longmapsto V_t f(x)$, $x \in \overline{\mathcal{O}}$, is negative definite on $C_+(\overline{\mathcal{O}})$ (:= the set of all positive continuous functions on $\overline{\mathcal{O}}$). We use essentially an approximating process in solving (1.1) and a negative definiteness property of the mapping $-\beta$.

The last step is to follow the so called semigroup approach in order to construct the claimed measure-valued branching process; see [19,14,13,15,16], and [3].

We can describe the infinitesimal generator of the branching process X, which shows that it behaves as the linear semiflow $t \longmapsto \mu \circ P_t$, $t \geqslant 0$, $\mu \in M(\mathcal{O})$ (:= the set of all positive finite measures on $\mathcal{O}$), where