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Finiteness of polygonal relative equilibria for generalised quasi-homogeneous n-body problems and n-body problems in spaces of constant curvature

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A B S T R A C T

We prove for generalisations of quasi-homogeneous n-body problems with centre of mass zero and n-body problems in spaces of negative constant Gaussian curvature that if the masses and rotation are fixed, there exists, for every order of the masses, at most one equivalence class of relative equilibria for which the point masses lie on a circle, as well as that there exists, for every order of the masses, at most one equivalence class of relative equilibria for which all but one of the point masses lie on a circle and rotate around the remaining point mass. The method of proof is a generalised version of a proof by J.M. Cors, G.R. Hall and G.E. Roberts on the uniqueness of co-circular central configurations for power-law potentials.

1. Introduction

By n-body problems we mean problems where we are tasked with deducing the dynamics of n point masses described by a system of differential equations. The study of such problems has applications to various fields, including atomic physics, celestial mechanics, chemistry, crystallography, differential equations, dynamical systems, geometric mechanics, Lie groups and algebras, non-Euclidean and differential geometry, stability theory, the theory of polytopes and topology (see for example [1,2,16,17,19,27,53,61,60,59,66] and the references therein). The n-body problems that form the backbone of this paper are a generalisation of a class of quasi-homogeneous n-body problems, which we will call generalised n-body problems for short and the n-body problem in spaces of constant Gaussian curvature, or curved n-body problem for short. By the generalised n-body problem we mean the problem of finding the orbits of point masses $q_1, \ldots, q_n \in \mathbb{R}^2$ and respective masses $m_1 > 0, \ldots, m_n > 0$ determined by the system of differential equations

$$\ddot{q}_i = \sum_{j=1,\ j\neq i}^{n} m_j (q_j - q_i) f\left(\|q_j - q_i\|\right), \tag{1.1}$$

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where $\|\cdot\|$ is the Euclidean norm, f is a positive valued scalar function and $xf(x)$ is a decreasing, differentiable function. Our definition of generalised n-body problems thus includes a large subset of quasi-homogeneous n-body problems, which are problems with $f(x) = Ax^{-a} + Bx^{-b}$, where $A, B \in \mathbb{R}$ and $0 \leq a < b$, which include problems studied in fields such as celestial mechanics, crystallography, chemistry and electromagnetics (see for example [8–10,13,12,14,11,25–27,33] and [49,50,52,51]).

By the n-body problem in spaces of constant Gaussian curvature, we mean the problem of finding the dynamics of point masses

$$p_1, \ldots, p_n \in \mathbb{M}^2_\sigma = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + \sigma x_3^2 = \sigma\},$$

where $\sigma = \pm 1$ and respective masses $\widehat{m}_1 > 0, \ldots, \widehat{m}_n > 0$, determined by the system of differential equations

$$\ddot{p}_i = \sum_{j=1,\ j\neq i}^{n} \frac{\widehat{m}_j (p_j - \sigma(p_i \odot p_j)p_i)}{(\sigma - \sigma(p_i \odot p_j)^2)^{\frac{3}{2}}} - \sigma(\dot{p}_i \odot \dot{p}_i)p_i,\ \ i \in \{1, \ldots, n\}, \tag{1.2}$$

where for $x,\ y \in \mathbb{M}^2_\sigma$ the product $\cdot \odot \cdot$ is defined as

$$x \odot y = x_1 y_1 + x_2 y_2 + \sigma x_3 y_3.$$

The curved n-body problem generalises the classical, or Newtonian n-body problem ($f(x) = x^{-\frac{3}{2}}$ in (1.1)) to spaces of constant Gaussian curvature (i.e. spheres and hyperboloids) and goes for the two body case back to the 1830s, (see [6] and [43]), followed by [57,58,36–38,40–42,39], but it was not until a revolution took place with the papers [30,28,29] by Diacu, Pérez-Chavela and Santoprete in which the successful study of n-body problems in spaces of constant Gaussian curvature for the case that $n \geq 2$ was established. After this breakthrough, further results for the $n \geq 2$ case were then obtained in [7,15,16,18,17,20–24] and [62–65]. See [15,16,18,17] and [21] for a detailed historical overview of the development of the curved n-body problem. In this paper we will only consider the negative constant curvature case, i.e. the case $\sigma = -1$.

For these two types of n-body problems we will prove results regarding the finiteness of relative equilibrium solutions of (1.1) and (1.2), which are solutions of (1.1), or (1.2), for which the configuration of the point masses stays fixed in shape and size over time. Specifically:

We will call $q_1, \ldots, q_n \in \mathbb{R}^2$ a *relative equilibrium* of (1.1) if $q_i(t) = T(At)(Q_i - Q_M) + Q_M$, $i \in \{1, \ldots, n\}$, where $Q_i \in \mathbb{R}^2$, $A \in \mathbb{R}_{>0}$ are constant,

$$T(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

is a 2×2 rotation matrix and

$$Q_M = \frac{1}{M} \sum_{k=1}^{n} m_k Q_k$$

is the *center of mass* with $M = \sum_{k=1}^{n} m_k$. If the q_i lie on a circle with the origin at its center, we will call $q_1, \ldots, q_n$ a polygonal relative equilibrium solution of (1.1). If all but one of the masses form a polygon with the origin at its center, with the remaining mass at the origin, then we will call such a relative equilibrium a polygonal relative equilibrium with center zero of (1.1) for short.

Following the example of [30,28,29] by Diacu, Pérez-Chavela and Santoprete, we will call $p_1, \ldots, p_n \in \mathbb{M}^2_\sigma$ a polygonal relative equilibrium of (1.2) if

$$p_i(t) = \begin{pmatrix} T(Bt)P_i \\ z \end{pmatrix},$$