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Entire solutions of fully nonlinear elliptic equations with a superlinear gradient term

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A B S T R A C T

In this paper we consider second order fully nonlinear operators with an additive superlinear gradient term. Like in the pioneering paper of Brezis for the semilinear case, we obtain the existence of entire viscosity solutions, defined in all the space, without assuming global bounds. A uniqueness result is also obtained for special gradient terms, subject to a convexity/concavity type assumption where superlinearity is essential and has to be handled in a different way from the linear case.

1. Introduction

We are interested in existence and uniqueness of solutions in $\mathbb{R}^n$ of fully nonlinear second order uniformly elliptic equations having superlinear growth in u and Du.

Before describing our results, let us recall some existing ones related to our work. In the pioneering work [4] Brezis considered *entire* solutions, i.e., defined in the whole space, of the semilinear elliptic problem

$$\Delta u - |u|^{s-1}u = f(x), \quad s > 1, \tag{1.1}$$

showing that it is well-posed in $\mathcal{D}'(\mathbb{R}^n)$ without prescribing conditions at infinity on f and u. The uniquely existence of solutions $u \in L^s_{\text{loc}}(\mathbb{R}^n)$ is proved by assuming only $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Moreover $u \geq 0$ a.e. if $f \leq 0$ a.e. in $\mathbb{R}^n$.

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This result was generalized by Gallouët and Morel [13] to the case

$$\Delta u - g(u) = f(x) \tag{1.2}$$

with a suitable function g satisfying an integral condition of Osserman type. Analogous results have been proved by Boccardo, Gallouët and Vázquez [3] and Leoni [20], respectively for equations (1.1) and (1.2), in the case where the Laplace operator is replaced by the p-Laplace operator and similar operators in divergent form.

In a recent paper [1] Alarcón, García-Melián and Quaas proved various results of existence and uniqueness of distributional and strong solutions of equation (1.1) in Sobolev spaces when a superlinear gradient term $|Du|^m$ is subtracted on the left-hand side of (1.1). On the other side, the result of Brezis has been extended by Esteban, Felmer and Quaas [10] to the class of fully nonlinear uniformly elliptic problems

$$F(D^2u) - |u|^{s-1}u = f(x) \quad \text{in } \mathbb{R}^n \tag{1.3}$$

where $f \in L^n_{\text{loc}}(\mathbb{R}^n)$ and the solution u is intended in the L^n-viscosity sense.

In [12] a further generalization has been provided by Galise and Vitolo to uniformly elliptic equation where the operator also depends on x and on the gradient. Positive entire solutions as well as subsolutions have been recently investigated by Felmer, Quaas and Sirakov [11]. Information on the behavior at infinity of positive entire solutions can be found in [22]. Conditions of Keller–Osserman type for the existence of entire subsolutions of degenerate elliptic equations have been established very recently by Capuzzo Dolcetta, Leoni and Vitolo [6,7].

In [12], in particular, following the original ideas of Brezis, combined with viscosity type arguments, the authors prove the existence of entire solutions of the uniformly elliptic equation

$$F(x, D^2u) + H(Du) - |u|^{s-1}u = f(x), \tag{1.4}$$

where $F(\cdot, X)$ is merely a measurable function for every X, the Hamiltonian $H : \mathbb{R}^n \mapsto \mathbb{R}$ is Lipschitz-continuous in the gradient variable, s is a real number strictly larger than 1 and $f \in L^n_{\text{loc}}(\mathbb{R}^n)$. Concerning the uniqueness, it is a remarkable fact that if the principal part F is independent on x, the well-posedness of (1.4) is ensured assuming only the continuity of the datum f, while in the general case further assumptions are needed in order to control the oscillation in the x-variable and the regularity of the solutions.

In this paper, we propose to study the well-posedness in the whole space for the equation

$$F(x, D^2u) + H(x, Du) - |u|^{s-1}u = f(x), \tag{1.5}$$

where $H(x, \cdot)$ may have a superlinear growth in the first derivative. It is worth recalling at this step that subquadratic Hamiltonians arise in optimal control problems for suitable models of running cost function in the framework of Bellman dynamic programming, as discussed by Lasry and Lions [19].

In (1.5), we will assume the following. The second order term

$$F \text{ is } (\lambda, \Lambda)\text{-uniformly elliptic and } F(x, 0) = 0 \text{ a.e. in } \mathbb{R}^n, \tag{1.6}$$

see (2.2) for a definition. As regards $H : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$, we will assume

$$H(x, 0) = 0 \quad \text{a.e. in } \mathbb{R}^n \tag{1.7}$$

and there exist $\gamma_1, \gamma_m > 0$, $m > 1$ such that

$$|H(x, p) - H(x, q)| \le \left(\gamma_1 + \gamma_m(|p|^{m-1} + |q|^{m-1})\right) |p - q| \tag{1.8}$$

for p, $q \in \mathbb{R}^n$ and a.e. $x \in \mathbb{R}^n$.