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Asymptotic behaviors of solutions to quasilinear elliptic equations with Hardy potential

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Optimal estimates on asymptotic behaviors of weak solutions both at the origin and at the infinity are obtained to the following quasilinear elliptic equations

$$-\Delta_p u - \frac{\mu}{|x|^p}|u|^{p-2}u + m|u|^{p-2}u = f(u), \qquad x \in \mathbb{R}^N,$$

where $1 < p < N$, $0 \le \mu < ((N-p)/p)^p$, $m > 0$ and f is a continuous function.

Contents

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1. Introduction and main results

In this note, we study asymptotic behaviors of weak solutions to the following quasilinear elliptic equations

$$-\Delta_p u - \frac{\mu}{|x|^p}|u|^{p-2}u + m|u|^{p-2}u = f(u), \qquad x \in \mathbb{R}^N, \tag{1.1}$$

where $1 < p < N$, $0 \le \mu < \bar{\mu} = ((N-p)/p)^p$, $m > 0$,

$$\Delta_p u = \sum_{i=1}^{N} \partial_{x_i}(|\nabla u|^{p-2}\partial_{x_i} u), \qquad \nabla u = (\partial_{x_1} u, \cdots, \partial_{x_N} u),$$

is the p-Laplacian operator and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function denoted by $f \in C(\mathbb{R})$. In addition, we assume throughout the paper that f satisfies that

$$\limsup_{t\to 0} \frac{|f(t)|}{|t|^{q-1}} \le A < \infty \tag{1.2}$$

for some $q > p$, and that

$$\limsup_{|t|\to\infty} \frac{|f(t)|}{|t|^{p^*-1}} \le A < \infty \tag{1.3}$$

with $p^* = Np/(N-p)$, where $A > 0$ is a constant.

Equation (1.1) is the Euler–Lagrange equation of the energy functional $E : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$E(u) = \frac{1}{p}\int_{\mathbb{R}^N} \left(|\nabla u|^p - \frac{\mu}{|x|^p}|u|^p + m|u|^p\right) - \int_{\mathbb{R}^N} F(u), \qquad u \in W^{1,p}(\mathbb{R}^N),$$

where F is given by $F(t) = \int_0^t f$ for $t \in \mathbb{R}$ and $W^{1,p}(\mathbb{R}^N)$ is the Banach space of weakly differentiable functions $u : \mathbb{R}^N \to \mathbb{R}$ such that the norm

$$\|u\|_{1,p,\mathbb{R}^N} = \left(\int_{\mathbb{R}^N} |u|^p\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^N} |\nabla u|^p\right)^{\frac{1}{p}}$$

is finite.

All of the integrals in energy functional E are well defined, due to the Sobolev inequality

$$C\left(\int_{\mathbb{R}^N} |\varphi|^{p^*}\right)^{\frac{p}{p^*}} \le \int_{\mathbb{R}^N} |\nabla\varphi|^p, \qquad \forall\, \varphi \in W^{1,p}(\mathbb{R}^N),$$

where $C = C(N,p) > 0$, and due to the Hardy inequality (see [3, Lemma 1.1])

$$\left(\frac{N-p}{p}\right)^p \int_{\mathbb{R}^N} \frac{|\varphi|^p}{|x|^p} \le \int_{\mathbb{R}^N} |\nabla\varphi|^p, \qquad \forall\, \varphi \in W^{1,p}(\mathbb{R}^N), \tag{1.4}$$

and due to the assumptions (1.2) and (1.3), which imply that

$$|F(t)| \le C|t|^p + C|t|^{p^*}, \qquad \forall\, t \in \mathbb{R},$$

for some positive constant C.