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Minimal surfaces in a Randers sphere with the rotational Killing vector field ☆

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ABSTRACT

The minimal surfaces in Finsler geometry with respect to the Busemann–Hausdorff measure and the Holmes–Thompson measure are called BH-minimal and HT-minimal surfaces, respectively. Let (p^1, p^2, p^3, p^4) be the coordinates of R^4 and $(S^3, \tilde{F})$ be a Randers sphere of flag curvature $\mathbf{K} = 1$ with the navigation data $(\tilde{h}, \tilde{W})$, where $\tilde{h}$ is the standard sphere metric and $\tilde{W} = \varepsilon(0, 0, -p^4, p^3)$, $0 < \varepsilon < 1$, is a Killing vector field. In this paper, we study the rotationally invariant minimal surface in $(S^3, \tilde{F})$ generated by rotating the curve $(x(s), y(s), z(s), 0)$ in the upper half sphere of S^2 around the p^1p^2-plane, $s \in R$. We first show that such a rotational BH-minimal surface in $(S^3, \tilde{F})$ is either a great 2-sphere or the catenoid in $(S^3, \tilde{h})$. Then we give a classification of the rotational HT-minimal surfaces, where we use the *angle data* to analyze the solutions of the system of ODE that characterizes the HT-minimality and prove that, such a rotational HT-minimal surface must be a great 2-sphere, an HT-minimal torus, or a rotational surface of unduloid type. As a special case, we obtain a distinguished embedded compact HT-minimal torus depending on ε. The completeness of these surfaces is also studied.

1. Introduction

In classical differential geometry, rotational surfaces with an additional property (constant mean curvature, constant Gaussian curvature, minimal, etc.) are probably the simplest surfaces having the specified property, and provide a good test-ground for various conjectures. Similar to the minimal surface theory in Riemannian geometry, it may be interesting to understand the minimal surfaces in the Finsler space forms (i.e., simply connected Finsler manifolds of constant flag curvature). A Randers metric is defined as the sum of a Riemannian metric and a 1-form, which was firstly introduced in the research on the general relativity and has been widely applied in many areas of natural science such as biology, physics and psychology, etc.

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As the simplest but the most important case, the Randers space forms have been classified by using the Zermelo navigation in [2].

As in Riemannian space forms, the global behavior of the rotationally invariant minimal surfaces in various Randers space forms should be quite different and they deserve to be well studied. Up to now, it has been well understood only in some Randers space forms of vanishing flag curvature, for example the Euclidean 3-space with perturbations of a translation and a rotational Killing vector field ([5,9,10]). For the positively curved Randers space forms, the local expression of the rotationally invariant minimal surface in a Randers 3-sphere with the perturbation of the Reeb vector field was given in [4].

The rotationally invariant minimal surfaces are usually characterized by an ODE that sometimes can be explicitly solved (for instance [5]). But most of the time it is difficult to be completely solved, and in this case we turn to analyze the solutions of the ODE to figure out the global pictures of the minimal surfaces (for instance in [10]), after all our aim is to understand the global properties of the minimal surfaces well.

It is well known that the positively curved Randers space forms are modeled on a standard sphere with Killing vector fields ([2]). Except for the Reeb vector field, we have another distinguished Killing vector field which is a rotation and its perturbation also generates a Randers sphere of flag curvature $\mathbf{K} = 1$. The main purpose of this paper is to study a rotationally invariant minimal surface in such a Randers 3-sphere perturbed by a rotational Killing field. By a detailed analysis of the system of ODE, we give a classification of such a surface which is minimal under the Busemann–Hausdorff measure and Holmes–Thompson measure respectively. (See Theorem 3.5.) Especially, an interesting HT-minimal torus and the HT-minimal surface of unduloid type (see Fig. 2) are obtained.

2. Preliminaries

A Finsler metric on M is a continuous function $F : TM \to [0, +\infty)$ satisfying: (i) F is smooth on $TM \setminus \{0\}$; (ii) $F(x, \lambda y) = \lambda F(x, y)$ for $(x, y) \in TM$ and any positive real number λ; (iii) The fundamental form $g := g_{ij} dx^i \otimes dx^j$ is positive definite on $TM \setminus \{0\}$, where $g_{ij} := \frac{1}{2}(\partial^2 F^2/\partial y^i \partial y^j)$. Here and from now on, we shall use the following convention of index ranges:

$$1 \leq i, j, \cdots \leq n; \quad 1 \leq \alpha, \beta, \cdots \leq n + p.$$

Einstein summation convention is also used throughout this paper. A smooth manifold endowed with a Finsler metric is called a Finsler manifold. A Finsler manifold (M, F) is said to be forward (geodesically) complete if every geodesic $\gamma(t)$, $a \leq t < b$, parametrized to be of constant Finslerian speed, can be extended to a geodesic defined on $a \leq t < +\infty$. Let $c : [a, b] \to M$ be a piecewise smooth curve. The length of c in F is defined as

$$L_F(c) := \int_a^b F\Big(c(t), \dot{c}(t)\Big) dt.$$

By using the Hopf–Rinow Theorem (see [1], pp. 168–172), one can deduce that M is forward complete if and only if the length of any divergent curve is unbounded.

Let $f : M^n \to (\tilde{M}^{n+p}, \tilde{F})$ be an isometric immersion in an $(n+p)$-dimensional Finsler manifold. In local coordinates, f can be written as $\tilde{x}^\alpha = f^\alpha(x^1, \cdots, x^n)$. In Finsler geometry, the Finsler metric can induce two well-known volume forms called Busemann–Hausdorff volume form and Holmes–Thompson volume form, and both of them reduce to the classical volume form when the Finsler metric is Riemannian. The induced metric $F := f^* \tilde{F}$ on M is also a Finsler metric, and at each point $x \in M$, the volume form dV_F of the induced metric F with respect to both two volume forms can be written in a uniform way