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Entire radially symmetric graphs with prescribed mean curvature in warped product spaces

Zonglao Zhang

College of Mathematics and Information Science, Wenzhou University, Wenzhou, Zhejiang, 325035, PR China

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ABSTRACT

This study investigates entire radially symmetric graphs in the warped product $M \times_f \mathbb{R}$, where M is a Riemannian manifold with a pole and $f : M \to \mathbb{R}^+$ is the warping function. The main results demonstrate the existence and uniqueness of entire radially symmetric graphs with prescribed mean curvature.

1. Introduction

The study of entire graphs with certain curvature properties plays an important role in classical differential geometry. In 1915, Bernstein published his well-known result that an entire minimal graph in three-dimensional Euclidean space $\mathbb{R}^3$ must be a plane (see [12]). In 1955, Heinz proved that if a graph in $\mathbb{R}^3$ defined by $z = \varphi(x, y)$, $x^2 + y^2 < R^2$, has constant mean curvature H, then $|H| \leq \frac{1}{R}$ (see [8]). This implies that the graph must be minimal if it is entire, and it should be a plane according to Bernstein's result. Subsequently, this topic was addressed by many mathematicians and it is still a very active research area. In the last few decades, many authors have extended the study of graphs from graphs in Euclidean space to those in hyperbolic space (e.g., see [9,10,13]), and even to those in warped product spaces (e.g., see [1–3,6,14,15]).

This study investigates entire radially symmetric graphs in the warped product $\overline{M} = M \times_f \mathbb{R}$, where M is a Riemannian manifold, $\mathbb{R}$ is the 1-dimensional Euclidean space with the standard metric, and f is a positive smooth function on M. We obtain some existence and uniqueness results for entire radially symmetric graphs with prescribed mean curvature. In the main results, the Riemannian manifold M is a radially symmetric manifold with a pole, or M is the n-dimensional hyperbolic space $\mathbb{H}^n$.

E-mail addresses: zonglao@163.com, zonglao@sohu.com.

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The remainder of this paper is organized as follows. In Section 2, we recall some notations and known results. In Section 3, we give our main results and their proofs.

2. Preliminaries

First, we recall the notation for the warped product (see [5,11]). Let (M, g) and $(N, \tilde{g})$ be Riemannian manifolds with metrics g and $\tilde{g}$, respectively, and f is a positive smooth function on M. Let π and $\tilde{\pi}$ be the projections of $M \times N$ onto M and N, respectively. The warped product $\overline{M} \equiv M \times_f N$ is the product manifold $M \times N$ equipped with the warped product metric

$$\bar{g} = \pi^*(g) + (f \circ \pi)^2 \tilde{\pi}^*(\tilde{g}).$$

We refer to f as the warping function of the warped product. If the warping function $f = 1$, then the warped product $M \times_f N$ is simply the usual Riemannian product $M \times N$.

In this study, we only consider the warped products $\overline{M} = M \times_f \mathbb{R}$, especially the case where $M = \mathbb{H}^n$. In this section and in the sequel, $\mathbb{H}^n$ denotes the n-dimensional hyperbolic space with sectional curvature -1 and $\mathbb{R}$ denotes the Euclidean line. It is interesting that $\mathbb{H}^n \times_f \mathbb{R} = \mathbb{H}^{n+1}$ when $f(x) = \cosh r(x)$ for any $x \in \mathbb{H}^n$, where $r(x)$ is the hyperbolic distance from x to some fixed point in $\mathbb{H}^n$ (see [10,16]). We refer to $\mathbb{H}^n \times_{\cosh r} \mathbb{R}$ as the warped product model of $\mathbb{H}^{n+1}$.

Let $\overline{M} = M \times_f \mathbb{R}$, $\varphi \in C^\infty(\Omega)$, where Ω is an open subset of M. We call

$$\Sigma = \{(x, \varphi(x)) \mid x \in \Omega\} \subset \overline{M}$$

the graph of the function φ in the warped product $M \times_f \mathbb{R}$. When $\Omega = M$, the graph Σ is said to be entire. The graph Σ of the function φ is a smooth submanifold of $\overline{M}$. Let ∇ and Δ be the gradient and Laplacian of M, respectively. The mean curvature H of the graph Σ (with respect to the upward pointing normal vector of Σ) is given by (see [14,15])

$$nH = \rho\Delta\varphi + (\nabla\varphi)\rho + \frac{\rho}{f}(\nabla\varphi)f, \tag{1}$$

where $n = \dim M$ and

$$\rho = \frac{1}{\sqrt{\frac{1}{f^2} + \|\nabla\varphi\|_M^2}}, \tag{2}$$

where $\|\nabla\varphi\|_M$ denotes the norm of the vector $\nabla\varphi$ in the metric of M.

Next, we recall the notation of a manifold with a pole (see [7]). Let M be a Riemannian manifold and $o \in M$. The point o is said to be a pole of M if the exponential map $\exp_o\colon T_oM \to M$ is a diffeomorphism, where T_oM denotes the tangent space of M at o. If M is a manifold with a pole o, we usually write (M, o) for M to clearly denote the pole under consideration.

Let (M, o) be a manifold with a pole. M is said to be radially symmetric around o if for any $\alpha, \tilde{\alpha} \in T_oM$ such that $|\alpha| = |\tilde{\alpha}|$, an isometry $\Psi : M \to M$ exists such that $\Psi(o) = o$ and $\Psi_*|_o(\alpha) = \tilde{\alpha}$, where $|\alpha|$ denotes the length of vector α and $\Psi_*|_o$ denotes the differential of Ψ at o. It should be noted that [7] used the term "weak model" to refer to our "radially symmetric manifold."

Let M be radially symmetric around o. For any $x \in M$, let $r(x) \equiv \mathrm{dist}(o, x)$, i.e., the geodesic distance from o to x. If $\phi(t)$ is a C^∞ function defined on an interval $I \subset [0, +\infty)$, and if we let $\varphi(x) \equiv \phi(r(x))$, then $\varphi(x)$ defines a function on $\Omega = \{x | r(x) \in I, x \in M\}$. We call $\varphi(x) = \phi(r(x))$ a radially symmetric function around o. For the sake of convenience, we often write $\varphi(x) = \varphi(r(x))$ for the radially symmetric